TEN-YEAR SERIES · WORKED SOLUTIONS

2024 A-Level H2 Mathematics Paper 1

12 questions · 100 marks

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Question 1 4 marks

Graphs and functions · asymptotes · turning points

APPROACH

three pieces of information give three equations in a, b, c: the vertical asymptote (denominator zero at $x = -\frac{1}{2}$), the turning point lying on the curve (substitute the point into the equation), and the turning point being stationary ($\frac{dy}{dx} = 0$ there). The third equation looks messy from the quotient rule, but evaluating its numerator at $x = -2$ turns out to cancel the terms in a, leaving a very short equation.

Since $y = \frac{x+1}{ax^2+bx+c}$ has a vertical asymptote at $x = -\frac{1}{2}$, the denominator vanishes there:

$$a\left(-\frac{1}{2}\right)^2 + b\left(-\frac{1}{2}\right) + c = 0$$
$$\Rightarrow \frac{a}{4} - \frac{b}{2} + c = 0$$

Multiplying by 4,

$$a - 2b + 4c = 0 \quad ...(1)$$

Since the turning point $\left(-2,\ -\frac{1}{9}\right)$ lies on the curve, substituting $x = -2$, $y = -\frac{1}{9}$:

$$\frac{-2+1}{a(4)+b(-2)+c} = -\frac{1}{9}$$
$$\Rightarrow \frac{-1}{4a-2b+c} = -\frac{1}{9}$$
$$\Rightarrow 4a - 2b + c = 9 \quad ...(2)$$

At a turning point $\frac{dy}{dx} = 0$. By the quotient rule,

$$\frac{dy}{dx} = \frac{\left(ax^2+bx+c\right)(1) - (x+1)(2ax+b)}{\left(ax^2+bx+c\right)^2}$$

The denominator is non-zero at $x = -2$ (since $x = -2 \neq -\frac{1}{2}$), so the numerator must vanish there. Evaluating the numerator's value at $x = -2$:

$$\left(ax^2+bx+c\right) - (x+1)(2ax+b)$$
$$= (4a-2b+c) - (-1)(-4a+b)$$
$$= 4a - 2b + c - 4a + b$$
$$= c - b$$

Note: the a-terms cancel because $-ax^2 - 2ax = -ax(x+2)$ vanishes at $x = -2$ for every a, so the turning-point condition reduces to

$$c - b = 0 \Rightarrow c = b \quad ...(3)$$

Substituting (3) into (1),

$$a - 2b + 4b = 0 \Rightarrow a + 2b = 0 \Rightarrow a = -2b$$

Substituting (3) and $a = -2b$ into (2),

$$4(-2b) - 2b + b = 9$$

$$\Rightarrow -9b = 9$$

$$\Rightarrow b = -1$$

$$a = -2(-1) = 2, \qquad c = b = -1$$

To check your answer: the denominator is $2x^2 - x - 1 = (2x+1)(x-1)$, giving the asymptote $x = -\frac{1}{2}$ ✓. At $x = -2$: $y = \frac{-1}{8+2-1} = -\frac{1}{9}$ ✓, and $\left.\frac{dy}{dx}\right|_{x=-2} = 0$ ✓.

$\therefore a = 2,\ b = -1,\ c = -1.$

Question 2 3 marks

Vectors · cross product · scalar product

Since $\mathbf{a}$ is a unit vector, $|\mathbf{a}| = 1$.

Since $\mathbf{b} \cdot \mathbf{b} = 9$, $|\mathbf{b}|^2 = 9 \Rightarrow |\mathbf{b}| = 3$.

Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$. Using $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta$:

$$3 = (1)(3)\sin\theta$$

$$\Rightarrow \sin\theta = 1$$

To check your answer: $|\mathbf{b}| = \sqrt{9} = 3$, and $(1)(3)(1) = 3 = |\mathbf{a} \times \mathbf{b}|$. ✓

Since $0 \le \theta \le 180°$, $\sin\theta = 1 \Rightarrow \theta = 90°$.

$\therefore$ since $\theta = 90°$, $\mathbf{a}$ and $\mathbf{b}$ are perpendicular. (shown)

Question 3 4 marks

Complex numbers · conjugate roots

Part (a) 4 marks

$$a,\ b,\ c \text{ are all real}$$

Explanation: for a polynomial with real coefficients, non-real roots occur in conjugate pairs. Since w is a (non-real) root, the third root is w^* only if the coefficients a, b, c are all real.

Part (b)

The three roots are 3, $w = -1 + 2i$ and $w^* = -1 - 2i$.

$$w + w^* = (-1 + 2i) + (-1 - 2i) = -2$$

$$w \cdot w^* = |w|^2 = (-1)^2 + 2^2 = 5$$

Since $z^3 + az^2 + bz + c \equiv (z-3)(z-w)\left(z-w^*\right)$, and the quadratic factor with roots w, w^* is $z^2 - \left(w + w^*\right)z + w \cdot w^*$:

$$(z-3)(z-w)\left(z-w^*\right)$$
$$=(z-3)\left(z^2-\left(w+w^*\right)z+w\cdot w^*\right)$$
$$=(z-3)\left(z^2+2z+5\right)$$

Expanding,

$$(z-3)\left(z^2+2z+5\right)$$
$$=z^3+2z^2+5z-3z^2-6z-15$$
$$=z^3-z^2-z-15$$

To check your answer: substituting $z=3$ into z^3-z^2-z-15 gives $27-9-3-15=0$, confirming 3 is indeed a root. ✓

Comparing with z^3+az^2+bz+c:

$$a=-1, \qquad b=-1, \qquad c=-15$$

$\therefore a=-1,\ b=-1,\ c=-15.$

Question 4 6 marks

Inequalities · sign diagram · modulus

Part (a) 4 marks

APPROACH

The sign of $x+2$ and of x is unknown, so we cannot cross-multiply directly — multiplying by a possibly negative quantity would flip the inequality without our knowing it. Instead we bring everything to one side and read the sign off a single combined fraction.

Note first that $x\neq -2,\ 0$, since both denominators must be nonzero.

We bring all terms to one side and combine into a single fraction:

$$\frac{4}{x+2}-\frac{x-3}{x}\geq 0$$
$$\Rightarrow \frac{4x-(x-3)(x+2)}{x(x+2)}\geq 0$$

Expanding the numerator:

$$4x-(x-3)(x+2)$$
$$=4x-\left(x^2-x-6\right)$$
$$=-x^2+5x+6$$

So the inequality becomes

$$\frac{-x^2+5x+6}{x(x+2)}\geq 0$$

Multiplying both sides by -1 flips the inequality:

$$\frac{x^2-5x-6}{x(x+2)} \leq 0$$
$$\Rightarrow \frac{(x-6)(x+1)}{x(x+2)} \leq 0$$

The critical values are $x = -2,\ -1,\ 0,\ 6$. Using a sign table across these:

Interval	$(-\infty, -2)$	$(-2, -1)$	$(-1, 0)$	$(0, 6)$	$(6, \infty)$
Sign	$+$	-	$+$	-	$+$

The expression is ≤ 0 on $(-2, -1]$ and $(0, 6]$; the endpoints -1 and 6 are included since the expression is zero there, while $x = -2, 0$ stay excluded since the expression is undefined there.

To check your answer: at $x = 1$, LHS $= \frac{4}{3} \approx 1.33$ and RHS $= \frac{-2}{1} = -2$, so $1.33 \geq -2$ ✓ (correctly in the solution set); at $x = -0.5$, LHS $= \frac{4}{1.5} \approx 2.67$ and RHS $= \frac{-3.5}{-0.5} = 7$, so $2.67 \geq 7$ is false ✓ (correctly excluded).

$\therefore -2 < x \leq -1$ or $0 < x \leq 6$.

Part (b) 2 marks

$$-6 \leq x < 0 \quad \text{or} \quad 0 < x \leq 6$$

Explanation: Part (a) shows that $\frac{4}{t+2} \geq \frac{t-3}{t}$ holds exactly when $-2 < t \leq -1$ or $0 < t \leq 6$. Putting $t = |x|$ reduces the given inequality to exactly this condition on $|x|$:

$$-2 < |x| \leq -1 \quad \text{or} \quad 0 < |x| \leq 6$$

Since $|x| \geq 0$ always, $|x| \leq -1$ is impossible, so the first alternative never holds. Only $0 < |x| \leq 6$ remains, i.e. $x \neq 0$ and $-6 \leq x \leq 6$.

$\therefore -6 \leq x < 0$ or $0 < x \leq 6$.

Question 5 7 marks

Sequences and series · sum to infinity

Part (a) 3 marks

Since $S_\infty = 18$ and $u_3 = \frac{8}{3}$, we use the standard geometric series formulas:

$$\frac{a}{1-r} = 18 \Rightarrow a = 18(1-r) \quad ...(1)$$
$$ar^2 = \frac{8}{3} \quad ...(2)$$

Substituting (1) into (2):

$$18(1-r)r^2 = \frac{8}{3}$$

Multiplying both sides by 3, expanding, and rearranging:

$$54(1-r)r^2 = 8$$
$$\Rightarrow 54r^2 - 54r^3 = 8$$
$$\Rightarrow -54r^3 + 54r^2 - 8 = 0$$

Dividing throughout by -2:

$$27r^3 - 27r^2 + 4 = 0$$

$\therefore 27r^3 - 27r^2 + 4 = 0$ (shown).

Part (b) 1 mark

Testing $r = -\frac{1}{3}$:

$$27\left(-\frac{1}{27}\right) - 27\left(\frac{1}{9}\right) + 4 = -1 - 3 + 4 = 0,$$

so $(3r+1)$ is a factor. Dividing $27r^3 - 27r^2 + 4$ by $(3r+1)$:

$$27r^3 - 27r^2 + 4 = (3r+1)\left(9r^2 - 12r + 4\right)$$
$$= (3r+1)(3r-2)^2$$

The roots are $r = -\frac{1}{3}$ or $r = \frac{2}{3}$ (repeated). Since $r < 0$ is given, $r = \frac{2}{3}$ is rejected.

$$r = -\frac{1}{3}$$

From (1), $a = 18\left(1 - \left(-\frac{1}{3}\right)\right) = 18\left(\frac{4}{3}\right) = 24$.

To check your answer: third term $= ar^2 = 24\left(\frac{1}{9}\right) = \frac{8}{3}$ ✓

$\therefore r = -\frac{1}{3}$ and $a = 24$.

Part (c) 3 marks

APPROACH

$\sum_{r=k+1}^{\infty} u_r$ is the tail of the series after the kth term, so it is most easily written as $S_\infty - S_k$ rather than summed directly from $r = k+1$.

Let $S_k = \sum_{r=1}^{k} u_r = \frac{a\left(1-r^k\right)}{1-r}$ and let $T_k = \sum_{r=k+1}^{\infty} u_r$. Since $S_\infty = S_k + T_k$,

$$T_k = S_\infty - S_k$$
$$= \frac{a}{1-r} - \frac{a\left(1-r^k\right)}{1-r}$$
$$= \frac{ar^k}{1-r}$$

Using this, the given equation $80T_k = S_k$ becomes

$$80\left(\frac{ar^k}{1-r}\right) = \frac{a\left(1-r^k\right)}{1-r}$$

Since $\frac{a}{1-r} \neq 0$, we divide both sides by it:

$$80r^k = 1 - r^k$$
$$\Rightarrow 81r^k = 1$$
$$\Rightarrow r^k = \frac{1}{81}$$

Substituting $r = -\frac{1}{3}$:

$$\left(-\frac{1}{3}\right)^k = \frac{1}{81}$$

Since the right side is positive, k must be even, and then $\left(\frac{1}{3}\right)^k = \frac{1}{81} = \left(\frac{1}{3}\right)^4 \Rightarrow k = 4$, which is indeed even.

To check your answer: $\left(-\frac{1}{3}\right)^4 = \frac{1}{81}$ ✓; $S_4 = \frac{160}{9}$, $T_4 = 18 - \frac{160}{9} = \frac{2}{9}$, and $80 \times \frac{2}{9} = \frac{160}{9} = S_4$ ✓

$\therefore k = 4.$

Question 6 8 marks

Differentiation · tangent · area between curves

Part (a) 5 marks

Since $y = 2e^{x^3} + a$, we differentiate using the chain rule:

$$\frac{dy}{dx} = 2e^{x^3} \cdot 3x^2$$
$$= 6x^2e^{x^3}$$

At T $(x = 1)$:

$$y(1) = 2e + a, \qquad \left.\frac{dy}{dx}\right|_{x=1} = 6e$$

Using the point-gradient form for the tangent at $T = (1,\ 2e + a)$ with gradient $6e$:

$$y - (2e + a) = 6e(x - 1)$$
$$\Rightarrow y = 6ex - 6e + 2e + a$$
$$\Rightarrow y = 6ex - 4e + a$$
$$\Rightarrow y = e(6x - 4) + a$$

To check your answer: at $x = 1$, $y = e(6 - 4) + a = 2e + a$ ✓, matching the point on the curve.

$\therefore$ the tangent is $y = e(6x - 4) + a$, i.e. $p = 6,\ q = -4,\ r = 1.$

Part (b)(i) 1 mark

$$a = 4e$$

Explanation: since the tangent passes through the origin, we substitute $x = 0,\ y = 0$ into the tangent equation from (a):

$$\begin{aligned} 0 &= e\,(6(0) - 4) + a \\ &= -4e + a \end{aligned}$$

$\therefore\ a = 4e.$

Part (b)(ii) 2 marks

APPROACH

With $a = 4e$ fixed, we identify the region bounded by C, the line $x = 0$ and the tangent, check which curve lies on top over $0 \le x \le 1$, then integrate the difference. The term $\int 2e^{x^3}\,dx$ has no elementary antiderivative, so it is evaluated numerically by GC.

With $a = 4e$, the tangent becomes $y = e(6x - 4) + 4e = 6ex$, and the curve is $y = 2e^{x^3} + 4e$.

At $x = 0$: the curve gives $y = 2 + 4e \approx 12.87$, while the tangent gives $y = 0$. The curve lies above the tangent throughout $0 \le x \le 1$, meeting it only at T ($x = 1$).

Since the curve lies above the tangent, the area between C, the line $x = 0$ and the tangent is

$$\begin{aligned} \text{Area} &= \int_0^1 \left[\left(2e^{x^3} + 4e\right) - 6ex\right] dx \\ &= \int_0^1 2e^{x^3}\,dx + \int_0^1 4e\,dx - \int_0^1 6ex\,dx \end{aligned}$$

The last two integrals are elementary:

$$\int_0^1 4e\,dx = 4e, \qquad \int_0^1 6ex\,dx = 3e\left[x^2\right]_0^1 = 3e$$

The first integral has no elementary antiderivative, so we evaluate it numerically by GC:

$$\int_0^1 2e^{x^3}\,dx \approx 2.6838 \ \text{(by GC)}$$

Combining,

$$\begin{aligned} \text{Area} &\approx 2.6838 + 4e - 3e \\ &= 2.6838 + e \\ &\approx 5.4021 \end{aligned}$$

To check your answer: since $4e - 3e = e \approx 2.7183$, Area $\approx 2.6838 + 2.7183 \approx 5.4021$ ✓

$\therefore$ the area is $2.6838 + e \approx 5.4$ (1 d.p.).

Question 7 8 marks

Trigonometry · method of differences

Part (a) 2 marks

Since $f(r) = \cos(r\theta)$,

$$f(2r-1) - f(2r+1) = \cos\left((2r-1)\theta\right) - \cos\left((2r+1)\theta\right)$$

Using $\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$ with $A = (2r-1)\theta$, $B = (2r+1)\theta$:

$$\frac{A+B}{2} = \frac{(2r-1)\theta + (2r+1)\theta}{2} = 2r\theta, \qquad \frac{A-B}{2} = \frac{-2\theta}{2} = -\theta$$

Substituting,

$$\begin{aligned}\cos A - \cos B &= -2\sin(2r\theta)\sin(-\theta) \\ &= 2\sin\theta\sin(2r\theta)\end{aligned}$$

$\therefore f(2r-1) - f(2r+1) = 2\sin\theta\sin(2r\theta)$ (shown).

Part (b) 3 marks

APPROACH

Part (a) writes $2\sin\theta\sin(2r\theta)$ as a difference of two cosine terms. Summing this difference over $r = 1, \ldots, n$ telescopes, since each $\cos\left((2r+1)\theta\right)$ cancels the next term's $\cos\left((2(r+1)-1)\,\theta\right)$.

From (a), $2\sin\theta\sin(2r\theta) = \cos\left((2r-1)\theta\right) - \cos\left((2r+1)\theta\right)$. Summing from $r = 1$ to n:

$$2\sin\theta\sum_{r=1}^{n}\sin(2r\theta) = \sum_{r=1}^{n}\left[\cos\left((2r-1)\theta\right) - \cos\left((2r+1)\theta\right)\right]$$

$$= \left[\cos\theta - \cos 3\theta\right] + \left[\cos 3\theta - \cos 5\theta\right] + \cdots + \left[\cos\left((2n-1)\theta\right) - \cos\left((2n+1)\theta\right)\right]$$

$$= \cos\theta - \cos\left((2n+1)\theta\right)$$

Since $\sin\theta \neq 0$, we divide both sides by $2\sin\theta$:

$$\sum_{r=1}^{n}\sin(2r\theta) = \frac{\cos\theta - \cos\left((2n+1)\theta\right)}{2\sin\theta}$$

$\therefore \sum_{r=1}^{n}\sin(2r\theta) = \frac{\cos\theta - \cos((2n+1)\theta)}{2\sin\theta}$ (shown).

Part (c) 3 marks

APPROACH

We write $\sin\left(\frac{\pi r}{6}\right)\cos\left(\frac{\pi r}{6}\right)$ as $\frac{1}{2}\sin(2r\theta)$ with $\theta = \frac{\pi}{6}$, so that the sum matches part (b) exactly.

Using $\sin x\cos x = \frac{1}{2}\sin(2x)$:

$$\sin\left(\frac{\pi r}{6}\right)\cos\left(\frac{\pi r}{6}\right) = \frac{1}{2}\sin\left(\frac{\pi r}{3}\right)$$
$$= \frac{1}{2}\sin\left(2r\cdot\frac{\pi}{6}\right)$$

So

$$\sum_{r=1}^{n}\sin\left(\frac{\pi r}{6}\right)\cos\left(\frac{\pi r}{6}\right) = \frac{1}{2}\sum_{r=1}^{n}\sin\left(2r\cdot\frac{\pi}{6}\right)$$

This matches part (b) with $\theta = \frac{\pi}{6}$. Since $\sin\frac{\pi}{6} = \frac{1}{2} \neq 0$ and $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$,

$$\sum_{r=1}^{n}\sin\left(2r\cdot\frac{\pi}{6}\right) = \frac{\frac{\sqrt{3}}{2} - \cos\left(\frac{(2n+1)\pi}{6}\right)}{2\left(\frac{1}{2}\right)}$$
$$= \frac{\sqrt{3}}{2} - \cos\left(\frac{(2n+1)\pi}{6}\right)$$

so

$$\sum_{r=1}^{n}\sin\left(\frac{\pi r}{6}\right)\cos\left(\frac{\pi r}{6}\right) = \frac{\sqrt{3}}{4} - \frac{1}{2}\cos\left(\frac{(2n+1)\pi}{6}\right)$$

Since $2n+1$ is always odd, $\frac{(2n+1)\pi}{6}$ (period 12 in $2n+1$, i.e. period 6 in n) only ever takes the values $\frac{\pi}{6}, \frac{3\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{9\pi}{6}, \frac{11\pi}{6}$, so $\cos\left(\frac{(2n+1)\pi}{6}\right)$ takes only three distinct values: $\frac{\sqrt{3}}{2}$ (when $2n+1 \equiv 1, 11 \pmod{12}$), 0 (when $2n+1 \equiv 3, 9 \pmod{12}$), and $-\frac{\sqrt{3}}{2}$ (when $2n+1 \equiv 5, 7 \pmod{12}$).

To check your answer: at $n=1$, the sum is $\sin\frac{\pi}{6}\cos\frac{\pi}{6} = \frac{1}{2}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$, matching $\cos\left(\frac{3\pi}{6}\right) = 0 \Rightarrow \frac{\sqrt{3}}{4} - 0 = \frac{\sqrt{3}}{4}$ ✓

Substituting the three values of $\cos\left(\frac{(2n+1)\pi}{6}\right)$:

$$\cos(\cdots) = \frac{\sqrt{3}}{2} \Rightarrow \text{sum} = \frac{\sqrt{3}}{4} - \frac{1}{2}\cdot\frac{\sqrt{3}}{2} = 0$$
$$\cos(\cdots) = 0 \Rightarrow \text{sum} = \frac{\sqrt{3}}{4}$$
$$\cos(\cdots) = -\frac{\sqrt{3}}{2} \Rightarrow \text{sum} = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$$

$\therefore$ the three possible values are $0,\ \frac{\sqrt{3}}{4},\ \frac{\sqrt{3}}{2}$.

Question 8 9 marks

Maclaurin series · standard series

Part (a) 7 marks

APPROACH

since $y = \cos\left(1 - e^{2x}\right)$ is a composition, we find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ by the chain and product rules, then rewrite both derivatives in terms of y and $\frac{dy}{dx}$ to match the given form.

Let $u = 1 - e^{2x}$, so $y = \cos u$ and $\frac{du}{dx} = -2e^{2x}$. By the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= -\sin(u)\frac{du}{dx} \\ &= -\sin(u)\left(-2e^{2x}\right) \\ &= 2e^{2x}\sin u\end{aligned}$$

Differentiating again, by the product rule,

$$\begin{aligned}\frac{d^2y}{dx^2} &= 4e^{2x}\sin u + 2e^{2x}\cos(u)\frac{du}{dx} \\ &= 4e^{2x}\sin u + 2e^{2x}\cos(u)\left(-2e^{2x}\right) \\ &= 4e^{2x}\sin u - 4e^{4x}\cos u\end{aligned}$$

Since $\frac{dy}{dx} = 2e^{2x}\sin u$, we have $4e^{2x}\sin u = 2\frac{dy}{dx}$; and $\cos u = y$, so $4e^{4x}\cos u = 4ye^{4x}$. Substituting,

$$\begin{aligned}\frac{d^2y}{dx^2} &= 2\frac{dy}{dx} - 4ye^{4x} \\ &= 2\left(\frac{dy}{dx} - 2ye^{4x}\right)\end{aligned}$$

To check your answer: at $x = 0$, $u = 1 - 1 = 0$, so $y = \cos 0 = 1$ and $\frac{dy}{dx} = 2(1)\sin 0 = 0$; the relation gives $\frac{d^2y}{dx^2} = 2\left(0 - 2(1)(1)\right) = -4$, matching a direct computation at $x = 0$ ✓

$\therefore k = 2.$

Part (b)

APPROACH

the relation from (a) lets each derivative at $x = 0$ be found from the ones before it, without differentiating the original composite function directly.

At $x = 0$: $u = 1 - e^0 = 0$, so

$$y(0) = \cos 0 = 1, \qquad y'(0) = 2e^0\sin 0 = 0$$

Using $\frac{d^2y}{dx^2} = 2\left(\frac{dy}{dx} - 2ye^{4x}\right)$ at $x = 0$,

$$y''(0) = 2\left(y'(0) - 2y(0)e^0\right) = 2\left(0 - 2(1)(1)\right) = -4$$

Differentiating the relation once more with respect to x,

$$\frac{d^3y}{dx^3} = 2\left(\frac{d^2y}{dx^2} - 2\left[\frac{dy}{dx}e^{4x} + 4ye^{4x}\right]\right)$$

so at $x = 0$,

$$\begin{aligned}y'''(0) &= 2\left(-4 - 2\left[0 + 4(1)(1)\right]\right) \\ &= 2(-4 - 8) \\ &= -24\end{aligned}$$

Using the Maclaurin series $y = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \cdots$,

$$y = 1 + 0 - \frac{4}{2}x^2 - \frac{24}{6}x^3 + \cdots$$
$$= 1 - 2x^2 - 4x^3 + \cdots$$

To check your answer: expanding directly, $e^{2x} = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \cdots$, so $u = 1 - e^{2x} = -2x - 2x^2 - \frac{4}{3}x^3 - \cdots$ and $u^2 = 4x^2 + 8x^3 + \cdots$; then $\cos u \approx 1 - \frac{u^2}{2} = 1 - 2x^2 - 4x^3 + \cdots$ ✓

$\therefore \cos\left(1 - e^{2x}\right) \approx 1 - 2x^2 - 4x^3$ (first three non-zero terms).

Part (c) 2 marks

The first two non-zero terms of $\cos\left(1 - e^{2x}\right)$ found in (b) are 1 and $-2x^2$. Using the binomial series for $\frac{1}{\sqrt{a+bx^2}}$,

$$\frac{1}{\sqrt{a+bx^2}} = \frac{1}{\sqrt{a}}\left(1 + \frac{b}{a}x^2\right)^{-1/2}$$
$$= \frac{1}{\sqrt{a}}\left[1 - \frac{1}{2}\cdot\frac{b}{a}x^2 + \cdots\right]$$
$$= \frac{1}{\sqrt{a}} - \frac{b}{2a^{3/2}}x^2 + \cdots$$

Comparing the first two non-zero terms with $1 - 2x^2$,

$$\frac{1}{\sqrt{a}} = 1 \Rightarrow a = 1$$
$$-\frac{b}{2a^{3/2}} = -2 \Rightarrow \frac{b}{2} = 2 \Rightarrow b = 4$$

To check your answer: with $a = 1,\ b = 4$, $\frac{1}{\sqrt{1+4x^2}} = \left(1 + 4x^2\right)^{-1/2} = 1 - \frac{1}{2}\left(4x^2\right) + \cdots = 1 - 2x^2 + \cdots$, matching part (b) ✓

$\therefore a = 1$ and $b = 4$.

Question 9 10 marks

Parametric equations · tangent · area

Part (a) 2 marks

Since $x = 3t^2 + 2t$ and $y = t^2 + 2t^3$,

$$\frac{dx}{dt} = 6t + 2, \qquad \frac{dy}{dt} = 2t + 6t^2$$

Using $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$,

$$\frac{dy}{dx} = \frac{2t + 6t^2}{6t + 2}$$
$$= \frac{2t(1 + 3t)}{2(3t + 1)}$$

$$= t$$

To check your answer: at $t = 1$, $dx/dt = 8$ and $dy/dt = 8$, so $dy/dx = 1 = t$ ✓

$\therefore k = 1.$

Part (b) 3 marks

Since the tangent makes a $60°$ angle with the x-axis, its gradient is $\tan 60° = \sqrt{3}$. Using $\frac{dy}{dx} = t$ from part (a),

$$t = \sqrt{3}$$

Substituting into the parametric equations,

$$x = 3\left(\sqrt{3}\right)^2 + 2\sqrt{3} = 9 + 2\sqrt{3}$$
$$y = \left(\sqrt{3}\right)^2 + 2\left(\sqrt{3}\right)^3 = 3 + 6\sqrt{3}$$

$\therefore P = \left(9 + 2\sqrt{3},\ 3 + 6\sqrt{3}\right).$

Part (c) 5 marks

APPROACH

the boundary "the line parallel to the y-axis through Q" is $x = 16$, so the region runs from $x = 0$ (where C meets the origin) to $x = 16$; converting $\int y\,dx$ to a t-integral needs the value of t at Q.

We first find t at $Q = (16, 20)$ by solving $3t^2 + 2t = 16$:

$$3t^2 + 2t - 16 = 0$$
$$\Rightarrow t = \frac{-2 \pm \sqrt{4 + 192}}{6}$$
$$\Rightarrow t = \frac{-2 \pm 14}{6}$$
$$\Rightarrow t = 2 \ \text{ or } \ t = -\frac{8}{3}\ (\text{rej. as } t \geq 0)$$

so $t = 2$ at Q.

To check your answer: at $t = 2$, $y = (2)^2 + 2(2)^3 = 4 + 16 = 20$, matching $Q = (16, 20)$ ✓

At $t = 0$, $(x, y) = (0, 0)$. Since $\frac{dx}{dt} = 6t + 2 > 0$ and $\frac{dy}{dt} = 2t + 6t^2 \geq 0$ for $t \geq 0$, C rises from the origin to Q as t runs from 0 to 2, staying in the first quadrant. The required region is bounded by C, the x-axis and the line $x = 16$, so

$$\text{Area} = \int_0^{16} y\,dx = \int_0^2 y\,\frac{dx}{dt}\,dt = \int_0^2 \left(t^2 + 2t^3\right)(6t + 2)\,dt$$

Expanding the integrand,

$$\begin{aligned}&\left(t^2+2t^3\right)(6t+2)\\&=6t^3+2t^2+12t^4+4t^3\\&=12t^4+10t^3+2t^2\end{aligned}$$

Integrating,

$$\begin{aligned}\text{Area}&=\int_0^2\left(12t^4+10t^3+2t^2\right)dt\\&=\left[\frac{12}{5}t^5+\frac{5}{2}t^4+\frac{2}{3}t^3\right]_0^2\\&=\frac{12}{5}(32)+\frac{5}{2}(16)+\frac{2}{3}(8)\\&=\frac{384}{5}+40+\frac{16}{3}\end{aligned}$$

Using a common denominator of 15,

$$\text{Area}=\frac{1152}{15}+\frac{600}{15}+\frac{80}{15}=\frac{1832}{15}$$

To check your answer: $\frac{1832}{15}\approx 122.1$, a reasonable area given y reaches 20 over an x-range of 16 ✓

$\therefore$ the exact area is $\frac{1832}{15}$.

Question 10 14 marks

Integration · area · volume of revolution

Part (a) 3 marks

Since $u=\sqrt{4x+1}$, $u^2=4x+1$, so

$$x=\frac{u^2-1}{4}$$

and

$$x-2=\frac{u^2-1}{4}-2=\frac{u^2-9}{4}$$

Differentiating $u^2=4x+1$ with respect to x,

$$2u\frac{du}{dx}=4\Rightarrow\frac{du}{dx}=\frac{2}{u}\Rightarrow dx=\frac{u}{2}\,du$$

Substituting $\sqrt{4x+1}=u$, $x-2=\frac{u^2-9}{4}$ and $dx=\frac{u}{2}\,du$ into the integral,

$$\begin{aligned}\int\frac{\sqrt{4x+1}}{x-2}\,dx&=\int\frac{u}{\frac{u^2-9}{4}}\cdot\frac{u}{2}\,du\\&=\int\frac{4u}{u^2-9}\cdot\frac{u}{2}\,du\\&=\int\frac{2u^2}{u^2-9}\,du\end{aligned}$$

To check your answer: at $x = 6$, $u = \sqrt{25} = 5$, and $x - 2 = 4 = \frac{25-9}{4}$ ✓

$\therefore \int \frac{\sqrt{4x+1}}{x-2}\,dx = \int \frac{2u^2}{u^2-9}\,du$ (shown).

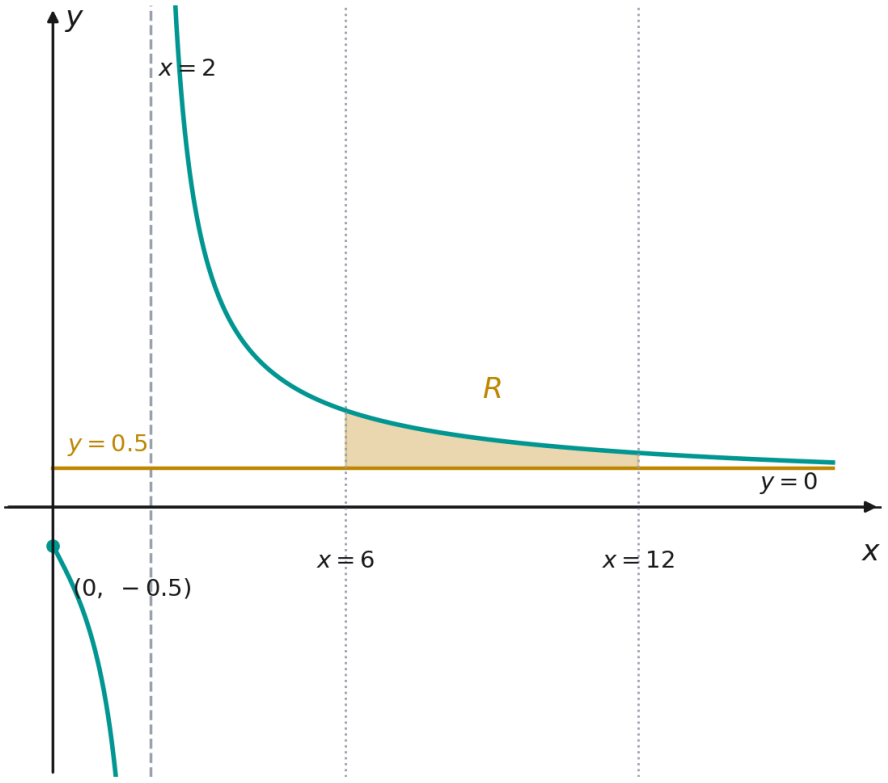
Part (b)(i) 3 marks

The curve $y = \frac{\sqrt{4x+1}}{x-2}$ has domain $x \geq 0,\ x \neq 2$.

At $x = 0$: $y = \frac{1}{-2} = -\frac{1}{2}$, giving the y-intercept $\left(0, -\frac{1}{2}\right)$. There is no x-intercept in the domain, since $\sqrt{4x+1} = 0$ would need $x = -\frac{1}{4}$, which is excluded.

The curve is undefined at $x = 2$, giving the vertical asymptote $x = 2$. As $x \to \infty$, $y \sim \frac{2\sqrt{x}}{x} = \frac{2}{\sqrt{x}} \to 0$, giving the horizontal asymptote $y = 0$.

The sketch below shows these features and the region R, bounded by the curve, the lines $x = 6$, $x = 12$ and $y = 0.5$.



Part (b)(ii) 5 marks

APPROACH

we split the area into $\int_6^{12} y\,dx$ minus the rectangle under $y = 0.5$, and evaluate the curve's integral using the substitution from part (a).

$$\text{Area} = \int_6^{12} \left(\frac{\sqrt{4x+1}}{x-2} - 0.5 \right) dx$$

From part (a), $\frac{2u^2}{u^2-9} = 2 + \frac{18}{u^2-9}$, and using the standard result $\int \frac{1}{u^2-9}\,du = \frac{1}{6} \ln \left| \frac{u-3}{u+3} \right| + C$,

$$\int \frac{2u^2}{u^2-9}\,du = 2u + 18\cdot\frac{1}{6}\ln\left|\frac{u-3}{u+3}\right| + C$$
$$= 2u + 3\ln\left|\frac{u-3}{u+3}\right| + C$$

At $x=6$, $u=\sqrt{25}=5$; at $x=12$, $u=\sqrt{49}=7$. So

$$\int_6^{12}\frac{\sqrt{4x+1}}{x-2}\,dx = \left[2u+3\ln\left|\frac{u-3}{u+3}\right|\right]_{u=5}^{u=7}$$
$$= \left(14+3\ln\frac{4}{10}\right) - \left(10+3\ln\frac{2}{8}\right)$$
$$= 4 + 3\ln\frac{2}{5} - 3\ln\frac{1}{4}$$
$$= 4 + 3\ln\left(\frac{2/5}{1/4}\right)$$
$$= 4 + 3\ln\frac{8}{5}$$

Subtracting $\int_6^{12} 0.5\,dx = 0.5(12-6) = 3$,

$$\text{Area} = \left(4+3\ln\frac{8}{5}\right) - 3 = 1 + 3\ln\frac{8}{5}$$

To check your answer: $1+3\ln\frac{8}{5} \approx 1+3(0.470) \approx 2.41$, matching a numerical estimate of the same integral ✓

$\therefore$ the exact area of R is $1+3\ln\frac{8}{5}$.

Part (b)(iii) 3 marks

APPROACH

since R is bounded above by the curve and below by $y=0.5$ (not by the x-axis), the solid of revolution is a washer — curve outer radius, line inner radius — not a simple disc.

By the washer method,

$$V = \pi\int_6^{12}\left[\left(\frac{\sqrt{4x+1}}{x-2}\right)^2 - (0.5)^2\right]dx = \pi\int_6^{12}\left[\frac{4x+1}{(x-2)^2} - 0.25\right]dx$$

Writing $4x+1 = 4(x-2)+9$,

$$\frac{4x+1}{(x-2)^2} = \frac{4(x-2)+9}{(x-2)^2}$$
$$= \frac{4}{x-2} + \frac{9}{(x-2)^2}$$

so

$$\int\left[\frac{4}{x-2} + \frac{9}{(x-2)^2} - 0.25\right]dx = 4\ln|x-2| - \frac{9}{x-2} - 0.25x + C$$

Evaluating at the limits:

$$\text{at } x = 12: \ 4\ln 10 - \frac{9}{10} - 3 = 4\ln 10 - 3.9$$
$$\text{at } x = 6: \ 4\ln 4 - \frac{9}{4} - 1.5 = 4\ln 4 - 3.75$$

Subtracting,

$$4\ln 10 - 3.9 - (4\ln 4 - 3.75) = 4\ln\frac{10}{4} - 0.15$$
$$= 4\ln\frac{5}{2} - 0.15$$

so

$$V = \pi\left(4\ln\frac{5}{2} - 0.15\right)$$

To check your answer: $4\ln 2.5 \approx 4(0.9163) = 3.665$; $3.665 - 0.15 = 3.515$; $\pi(3.515) \approx 11.04$ ✓

$\therefore V = \pi\left(4\ln\frac{5}{2} - 0.15\right) \approx 11.04$ (2 d.p.).

Question 11 12 marks

Differential equations · terminal velocity

Part (a)(i) 1 mark

$$\frac{dv}{dt} + \alpha v = 9.8$$

Explanation: since $v = \frac{dx}{dt}$, $\frac{dv}{dt} = \frac{d^2x}{dt^2}$. Substituting into the given equation $\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} = 9.8$ gives $\frac{dv}{dt} + \alpha v = 9.8$ directly.

Part (a)(ii) 1 mark

$$\alpha = 2$$

Explanation: substituting $\frac{dv}{dt} = 5$ and $v = 2.4$ into $\frac{dv}{dt} + \alpha v = 9.8$,

$$5 + 2.4\alpha = 9.8$$
$$\Rightarrow 2.4\alpha = 4.8$$
$$\Rightarrow \alpha = 2$$

Part (b) 5 marks

With $\alpha = 2$, the equation from (a)(i) becomes $\frac{dv}{dt} = 9.8 - 2v$. Separating variables,

$$\int \frac{dv}{9.8 - 2v} = \int dt$$
$$\Rightarrow -\frac{1}{2}\ln|9.8 - 2v| = t + c$$
$$\Rightarrow \ln|9.8 - 2v| = -2t + c'$$
$$\Rightarrow 9.8 - 2v = Ke^{-2t}, \text{ where } K = e^{c'}$$

Using the given initial condition $v = 0$ when $t = 0$,

$$9.8 - 0 = K \Rightarrow K = 9.8$$

so $9.8 - 2v = 9.8e^{-2t}$, giving

$$v = \frac{9.8}{2}\left(1 - e^{-2t}\right) = 4.9\left(1 - e^{-2t}\right)$$

To check your answer: at $t = 0$, $v = 4.9(1 - 1) = 0$, matching the initial condition. Also $\frac{dv}{dt} = 4.9\left(2e^{-2t}\right) = 9.8e^{-2t}$, and at $v = 0$, $t = 0$ this gives 9.8, consistent with $9.8 - 2(0)$ ✓

$\therefore A = 4.9$ and $B = -2$.

Part (c) 5 marks

APPROACH

we integrate $v(t)$ to get $x(t)$, set $x = 2000$, then note that for t of the size expected (hundreds of seconds), e^{-2t} is negligible, letting us solve the resulting equation without a GC.

Since $v = \frac{dx}{dt}$ and $x(0) = 0$ (the parachutist starts at the helicopter), integrating,

$$\begin{aligned} x(t) &= \int_0^t 4.9\left(1 - e^{-2t'}\right) dt' \\ &= 4.9\left[t' + \frac{1}{2}e^{-2t'}\right]_0^t \\ &= 4.9\left[\left(t + \frac{1}{2}e^{-2t}\right) - \left(0 + \frac{1}{2}\right)\right] \\ &= 4.9\left(t + \frac{1}{2}e^{-2t} - \frac{1}{2}\right) \end{aligned}$$

Setting $x = 2000$ (the height of the helicopter),

$$\begin{aligned} 4.9\left(t + \frac{1}{2}e^{-2t} - \frac{1}{2}\right) &= 2000 \\ \Rightarrow t + \frac{1}{2}e^{-2t} - \frac{1}{2} &= \frac{2000}{4.9} = \frac{20000}{49} \end{aligned}$$

For a parachutist falling for hundreds of seconds, t is large, so $e^{-2t} \approx 0$ and can be dropped:

$$t \approx \frac{20000}{49} + \frac{1}{2}$$

To check your answer: solving $t + \frac{1}{2}e^{-2t} - \frac{1}{2} = \frac{20000}{49}$ numerically without dropping e^{-2t} gives $t \approx 408.663$, agreeing with the approximation to many decimal places, since e^{-2t} is astronomically small at this t ✓

$\therefore$ the parachutist takes $t \approx 408.66$ seconds (2 d.p.) to reach the ground.

Question 12 15 marks

Vectors · closest approach · skew lines

Part (a)(i) 4 marks

> **APPROACH**
> The closest point on a line to an external point is where the vector from that point to the line is perpendicular to the line's direction vector — this gives one equation in λ.

The aircraft's path is $\mathbf{r} = \begin{pmatrix}4\\3\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\2\end{pmatrix}$, so a general point on it is $(4+\lambda,\ 3-\lambda,\ 1+2\lambda)$. The controllers sit 0.1 km above the base of the tower, i.e. at $(0, 0, 0.1)$.

The vector from the controllers to this general point is

$$\begin{pmatrix}4+\lambda\\3-\lambda\\1+2\lambda\end{pmatrix} - \begin{pmatrix}0\\0\\0.1\end{pmatrix}$$
$$= \begin{pmatrix}4+\lambda\\3-\lambda\\0.9+2\lambda\end{pmatrix}$$

At the closest point this vector is perpendicular to the direction vector $\begin{pmatrix}1\\-1\\2\end{pmatrix}$, so their dot product is zero:

$$\begin{pmatrix}4+\lambda\\3-\lambda\\0.9+2\lambda\end{pmatrix} \cdot \begin{pmatrix}1\\-1\\2\end{pmatrix} = 0$$
$$\Rightarrow (4+\lambda) - (3-\lambda) + 2(0.9+2\lambda) = 0$$
$$\Rightarrow 2.8 + 6\lambda = 0$$
$$\Rightarrow \lambda = -\frac{7}{15}$$

Substituting back into the path,

$$x = 4 - \frac{7}{15} = \frac{53}{15},\quad y = 3 + \frac{7}{15} = \frac{52}{15},\quad z = 1 - \frac{14}{15} = \frac{1}{15}$$

To check your answer: the vector from $(0,0,0.1)$ to $(\frac{53}{15}, \frac{52}{15}, \frac{1}{15})$ is $(\frac{53}{15}, \frac{52}{15}, -\frac{1}{30})$, and dotting with $(1,-1,2)$ gives $\frac{53}{15} - \frac{52}{15} - \frac{2}{30} = \frac{1}{15} - \frac{1}{15} = 0$ ✓

$\therefore$ the closest point is $(\frac{53}{15},\ \frac{52}{15},\ \frac{1}{15}) \approx (3.53,\ 3.47,\ 0.07)$.

Part (a)(ii) 2 marks

The distance from the controllers to this closest point is

$$d^2 = \left(\frac{53}{15}\right)^2 + \left(\frac{52}{15}\right)^2 + \left(\frac{1}{15} - 0.1\right)^2$$

Since $\frac{1}{15} - 0.1 = -\frac{1}{30}$,

$$\begin{aligned} d^2 &= \frac{2809}{225} + \frac{2704}{225} + \frac{1}{900} \\ &= \frac{11236}{900} + \frac{10816}{900} + \frac{1}{900} \\ &= \frac{22053}{900} \end{aligned}$$

$$d = \sqrt{\frac{22053}{900}} = \frac{\sqrt{22053}}{30} \approx 4.95 \text{ km (3 s.f.)}$$

Since $4.95 \text{ km} > 4 \text{ km}$, the aircraft does not come within 4 km of the controllers.

$\therefore$ the air traffic controllers will not see this aircraft.

Part (b) 3 marks

APPROACH

We convert the drone's Cartesian equation to vector form, then check whether the aircraft's and drone's paths ever intersect for a general k. Since k only appears in one of the three scalar equations, the values of λ, μ at any intersection are fixed by the other two equations regardless of k — this leaves a single value of k to exclude.

Writing the drone's path as $\frac{2x-3}{3} = y + 1 = \frac{z-k}{-1} = \mu$,

$$x = \frac{3\mu + 3}{2}, \quad y = \mu - 1, \quad z = k - \mu$$

so the drone's path is $\mathbf{r} = \begin{pmatrix} 1.5 \\ -1 \\ k \end{pmatrix} + \mu \begin{pmatrix} 1.5 \\ 1 \\ -1 \end{pmatrix}$, or, scaling the direction vector to clear decimals,

$\mathbf{r} = \begin{pmatrix} 1.5 \\ -1 \\ k \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$.

For the paths to intersect, some λ, μ must satisfy all three of

$$4 + \lambda = 1.5 + 1.5\mu \quad ...(1)$$
$$3 - \lambda = -1 + \mu \quad ...(2)$$
$$1 + 2\lambda = k - \mu \quad ...(3)$$

From (2), $\lambda = 4 - \mu$. Substituting into (1),

$$\begin{aligned} 4 + (4 - \mu) &= 1.5 + 1.5\mu \\ \Rightarrow 2.5\mu &= 6.5 \\ \Rightarrow \mu = 2.6, \quad \lambda &= 1.4 \end{aligned}$$

To check your answer: substituting $\mu = 2.6$ into (1) gives $\lambda = -2.5 + 1.5(2.6) = 1.4$, matching the value from (2) ✓

Substituting into (3), the paths intersect only when

$$1 + 2(1.4) = k - 2.6$$
$$\Rightarrow k = 6.4$$

Since λ, μ are pinned down by (1) and (2) alone, this is the only value of k at which an intersection occurs. As k is positive, for the paths to **not** intersect,

$\therefore k > 0$ and $k \neq 6.4$.

Part (c) 2 marks

The acute angle between the paths uses the direction vectors $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$:

$$\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} = 3 - 2 - 4 = -3$$
$$\left| \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right| = \sqrt{6}, \quad \left| \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} \right| = \sqrt{17}$$

Since the angle required is acute, we take the magnitude of the dot product:

$$\cos\theta = \frac{|-3|}{\sqrt{6}\sqrt{17}} = \frac{3}{\sqrt{102}}$$
$$\Rightarrow \theta = \cos^{-1}\left(\frac{3}{\sqrt{102}}\right)$$

To check your answer: $\sqrt{102} \approx 10.10$, so $\cos\theta \approx 0.297$ and $\theta \approx 72.7°$, which is acute as required ✓

$\therefore \theta \approx 72.7°$ (1 d.p.).

Part (d)(i) 4 marks

The vector between the two points is

$$\begin{pmatrix} 4.2 \\ 0.8 \\ 0.2 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.2 \\ -2.2 \\ -0.8 \end{pmatrix}$$
$$\text{distance} = \sqrt{(0.2)^2 + (-2.2)^2 + (-0.8)^2}$$
$$= \sqrt{0.04 + 4.84 + 0.64}$$
$$\Rightarrow \sqrt{5.52} = \frac{\sqrt{138}}{5}$$

$\therefore$ the distance is $\frac{\sqrt{138}}{5} \approx 2.35$ km (3 s.f.).

Part (d)(ii)

APPROACH

The shortest distance between two skew lines lies along a vector perpendicular to *both* direction vectors. It is enough to test the given connecting vector against just one direction vector: if it fails to be perpendicular to that one, it cannot be the shortest-distance vector.

The connecting vector found in (d)(i) is $\begin{pmatrix} 0.2 \\ -2.2 \\ -0.8 \end{pmatrix}$. Testing it against the aircraft's direction vector,

$$\begin{pmatrix} 0.2 \\ -2.2 \\ -0.8 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = 0.2 + 2.2 - 1.6 = 0.8$$

Since this dot product is $0.8 \neq 0$, the connecting vector is not perpendicular to the aircraft's direction vector.

$\therefore$ the distance found in (d)(i) is **not** the shortest distance between the two paths, since a vector giving the shortest distance must be perpendicular to both direction vectors, and this one is not even perpendicular to one of them.

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