TEN-YEAR SERIES · WORKED SOLUTIONS

2023 A-Level H2 Mathematics Paper 2

10 questions · 100 marks

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Question 1 6 marks

Equations and Inequalities · Inequalities

Part (a) 2 marks

Approach: both sides of $3|x-2| < |5-2x|$ are non-negative, so squaring removes the modulus signs without changing the inequality's direction — this avoids splitting into cases on the sign of $x-2$ and $5-2x$.

$$3|x-2| < |5-2x|$$

Squaring both sides,

$$\begin{aligned} 9(x-2)^2 &< (5-2x)^2 \\ \Rightarrow 9\left(x^2-4x+4\right) &< 25-20x+4x^2 \\ \Rightarrow 9x^2-36x+36 &< 4x^2-20x+25 \\ \Rightarrow 5x^2-16x+11 &< 0 \end{aligned}$$

Solving $5x^2-16x+11=0$ by the quadratic formula (discriminant $=(-16)^2-4(5)(11)=36$),

$$\begin{aligned} x &= \frac{16 \pm 6}{10} \\ &= \frac{11}{5} \text{ or } 1 \end{aligned}$$

Since the coefficient of x^2 is positive, $5x^2-16x+11<0$ between the roots.

To check your answer: at $x=1.5$ (inside the interval): $3|1.5-2|=1.5$ and $|5-3|=2$, and $1.5<2$. ✓ At $x=0$ (outside): $3|0-2|=6$ and $|5-0|=5$, and $6<5$ is false, correctly excluded. ✓

$\therefore 1 < x < \frac{11}{5}$.

Part (b) 4 marks

Approach: combine the given expression into a single fraction first — it factorises — then reuse that factorisation to solve the inequality, rather than expanding everything again from scratch.

$$\begin{aligned} &\frac{x+25}{x^2-4x-5}+3 \\ &= \frac{x+25+3\left(x^2-4x-5\right)}{x^2-4x-5} \\ &= \frac{3x^2-11x+10}{x^2-4x-5} \end{aligned}$$

Factorising the numerator (discriminant $=(-11)^2-4(3)(10)=1$) and the denominator,

$$\begin{aligned} 3x^2-11x+10 &= (x-2)(3x-5) \\ x^2-4x-5 &= (x-5)(x+1) \end{aligned}$$

so

$$\frac{x+25}{x^2-4x-5}+3 = \frac{(x-2)(3x-5)}{(x-5)(x+1)}$$

To check your answer: at $x = 0$: LHS $= \frac{25}{-5} + 3 = -2$; RHS $= \frac{(-2)(-5)}{(-5)(1)} = -2$. ✓

Hence, solving $\frac{(x-2)(3x-5)}{(x-5)(x+1)} > -3$: moving -3 to the left and combining into a single fraction,

$$\frac{(x-2)(3x-5)}{(x-5)(x+1)} + 3 > 0$$
$$\Rightarrow \frac{(x-2)(3x-5) + 3(x-5)(x+1)}{(x-5)(x+1)} > 0$$
$$\Rightarrow \frac{6x^2 - 23x - 5}{(x-5)(x+1)} > 0$$

Solving $6x^2 - 23x - 5 = 0$ by the quadratic formula (discriminant $= (-23)^2 - 4(6)(-5) = 649$, not a perfect square),

$$x = \frac{23 \pm \sqrt{649}}{12}$$

Since $\sqrt{649} \approx 25.5$, these roots are approximately -0.206 and 4.04. With the four critical values $x = -1$, $\frac{23-\sqrt{649}}{12}$, $\frac{23+\sqrt{649}}{12}$, 5 in increasing order, a sign chart gives $\frac{6x^2-23x-5}{(x-5)(x+1)} > 0$ on $(-\infty, -1)$, on $\left(\frac{23-\sqrt{649}}{12}, \frac{23+\sqrt{649}}{12}\right)$, and on $(5, \infty)$, with $x = -1$ and $x = 5$ excluded since the original expression is undefined there.

To check your answer: at $x = -2$ (in $(-\infty, -1)$): $\frac{6(4)-23(-2)-5}{(-7)(-1)} = \frac{65}{7} > 0$. ✓ At $x = -0.5$ (in the excluded gap): $\frac{6(0.25)+11.5-5}{(-5.5)(0.5)} = \frac{8}{-2.75} < 0$, correctly excluded. ✓

$\therefore x < -1$, or $\frac{23-\sqrt{649}}{12} < x < \frac{23+\sqrt{649}}{12}$, or $x > 5$.

Question 2 9 marks

Maclaurin Series · Deriving Maclaurin Series

Part (a) 3 marks

$$y = \ln(\sec x)$$

Differentiating using the chain rule,

$$\frac{dy}{dx} = \frac{\sec x \tan x}{\sec x}$$
$$= \tan x$$

Differentiating again,

$$\frac{d^2y}{dx^2} = \sec^2 x$$

Differentiating once more,

$$\frac{d^3y}{dx^3} = 2\sec x \cdot \sec x \tan x$$
$$= 2\sec^2 x \tan x$$

Since $\frac{d^2y}{dx^2} = \sec^2 x$ and $\frac{dy}{dx} = \tan x$,

$$\frac{d^3y}{dx^3} = 2\sec^2 x \tan x = 2\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right) \quad \text{(shown)}$$

Part (b) 4 marks

APPROACH

Part (a) gives y''' in terms of y'' and y', so rather than differentiating $2\sec^2 x \tan x$ from scratch, we differentiate the *relation* in (a) once more (product rule) to get y'''', then evaluate every derivative at $x = 0$.

Evaluating the derivatives at $x = 0$,

$$y(0) = \ln(\sec 0) = \ln 1 = 0, \quad y'(0) = \tan 0 = 0, \quad y''(0) = \sec^2(0) = 1$$

Using the result of (a), $y'''(0) = 2\,y''(0)\,y'(0) = 2(1)(0) = 0$.

Differentiating the relation in (a) once more,

$$\frac{d^4y}{dx^4} = 2\left[\frac{d^3y}{dx^3}\cdot\frac{dy}{dx} + \frac{d^2y}{dx^2}\cdot\frac{d^2y}{dx^2}\right]$$
$$= 2\left[\frac{d^3y}{dx^3}\cdot\frac{dy}{dx} + \left(\frac{d^2y}{dx^2}\right)^2\right]$$

At $x = 0$,

$$y''''(0) = 2\left[(0)(0) + (1)^2\right] = 2$$

Using the Maclaurin series $y = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y''''(0)}{4!}x^4 + \cdots$,

$$y = 0 + 0 + \frac{1}{2}x^2 + 0 + \frac{2}{24}x^4 + \cdots$$
$$= \frac{x^2}{2} + \frac{x^4}{12} + \cdots$$

To check your answer: at $x = 0.1$, the series gives $\frac{(0.1)^2}{2} + \frac{(0.1)^4}{12} \approx 0.005008$, close to the direct value $\ln(\sec 0.1) \approx 0.005005$, as expected for small x ✓

$\therefore \ln(\sec x) \approx \frac{x^2}{2} + \frac{x^4}{12}$.

Part (c) 2 marks

Putting $x = \frac{\pi}{4}$,

$$y = \ln\left(\sec\frac{\pi}{4}\right) = \ln\sqrt{2} = \frac{1}{2}\ln 2$$

Using the expansion from (b),

$$\frac{1}{2}\ln 2 \approx \frac{(\pi/4)^2}{2} + \frac{(\pi/4)^4}{12}$$
$$= \frac{\pi^2}{32} + \frac{\pi^4}{3072}$$
$$\ln 2 \approx \frac{\pi^2}{16} + \frac{\pi^4}{1536}$$

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To check your answer: $\frac{\pi^2}{16} + \frac{\pi^4}{1536} \approx 0.6803$, reasonably close to the true value $\ln 2 \approx 0.6931$, given that $x = \pi/4$ is not especially small ✓

$\therefore \ln 2 \approx \frac{\pi^2}{16} + \frac{\pi^4}{1536}$.

Part (d)

Using the expansion from (b),

$$\begin{aligned}\int_0^{\pi/10} \ln(\sec x)\, dx &\approx \int_0^{\pi/10} \left(\frac{x^2}{2} + \frac{x^4}{12}\right) dx \\ &= \left[\frac{x^3}{6} + \frac{x^5}{60}\right]_0^{\pi/10} \\ &= \frac{(\pi/10)^3}{6} + \frac{(\pi/10)^5}{60} \\ &= \frac{\pi^3}{6000} + \frac{\pi^5}{6\,000\,000}\end{aligned}$$

$\therefore \int_0^{\pi/10} \ln(\sec x)\, dx \approx 0.005219$ (4 s.f.).

Question 3 12 marks

Vectors · 3D Vector Geometry

Part (a) 2 marks

$$A = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ -2 \\ 8 \end{pmatrix}$$

We first find $\overrightarrow{AB}$.

$$\begin{aligned}\overrightarrow{AB} &= B - A \\ &= \begin{pmatrix} 1-(-1) \\ -2-2 \\ 8-5 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix}\end{aligned}$$

Since $\overrightarrow{BC} = 2\overrightarrow{AB}$,

$$\overrightarrow{BC} = 2\begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ -8 \\ 6 \end{pmatrix}$$

$$\begin{aligned}C &= B + \overrightarrow{BC} \\ &= \begin{pmatrix} 1+4 \\ -2-8 \\ 8+6 \end{pmatrix}\end{aligned}$$

$$= \begin{pmatrix} 5 \\ -10 \\ 14 \end{pmatrix}$$

To check your answer: $\overrightarrow{AC} = C - A = \begin{pmatrix} 6 \\ -12 \\ 9 \end{pmatrix}$, and $3\overrightarrow{AB} = \begin{pmatrix} 6 \\ -12 \\ 9 \end{pmatrix}$, matching, since $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AB} + 2\overrightarrow{AB} = 3\overrightarrow{AB}$ ✓

$\therefore C$ has position vector $\begin{pmatrix} 5 \\ -10 \\ 14 \end{pmatrix}$.

Part (b) 2 marks

Let $D = \begin{pmatrix} 1 \\ 2 \\ d \end{pmatrix}$. Since $\left|\overrightarrow{AD}\right| = \left|\overrightarrow{BD}\right|$, we first find both vectors.

$$\overrightarrow{AD} = D - A = \begin{pmatrix} 2 \\ 0 \\ d-5 \end{pmatrix}, \quad \overrightarrow{BD} = D - B = \begin{pmatrix} 0 \\ 4 \\ d-8 \end{pmatrix}$$

Equating the squared lengths,

$$4 + (d-5)^2 = 16 + (d-8)^2$$
$$\Rightarrow 4 + d^2 - 10d + 25 = 16 + d^2 - 16d + 64$$
$$\Rightarrow 29 - 10d = 80 - 16d$$
$$\Rightarrow 6d = 51$$
$$\Rightarrow d = \frac{17}{2}$$

To check your answer: at $d = 8.5$, $|AD|^2 = 4 + (8.5-5)^2 = 16.25$ and $|BD|^2 = 16 + (8.5-8)^2 = 16.25$, equal ✓

$\therefore d = \frac{17}{2}$.

Part (c) 3 marks

APPROACH

With D now fixed, $\angle$ADB is the angle between $\overrightarrow{DA}$ and $\overrightarrow{DB}$, found from their scalar product.

With $d = \frac{17}{2}$, $D = \begin{pmatrix} 1 \\ 2 \\ 8.5 \end{pmatrix}$.

$$\overrightarrow{DA} = A - D = \begin{pmatrix} -2 \\ 0 \\ -3.5 \end{pmatrix}, \quad \overrightarrow{DB} = B - D = \begin{pmatrix} 0 \\ -4 \\ -0.5 \end{pmatrix}$$

$$\overrightarrow{DA} \cdot \overrightarrow{DB} = (-2)(0) + (0)(-4) + (-3.5)(-0.5)$$
$$= 1.75$$

$$\left|\overrightarrow{DA}\right| = \sqrt{4+0+12.25} = \sqrt{16.25}, \quad \left|\overrightarrow{DB}\right| = \sqrt{0+16+0.25} = \sqrt{16.25}$$

$$\begin{aligned}\cos(\angle \text{ADB}) &= \frac{1.75}{\sqrt{16.25}\times\sqrt{16.25}} \\ &= \frac{1.75}{16.25} \\ &= \frac{7}{65}\end{aligned}$$

To check your answer: $\frac{7}{65} \approx 0.1077$, and $\cos^{-1}(0.1077) \approx 83.8°$, a plausible angle for a triangle with $|DA| = |DB|$ (isosceles, so the base angles at A and B are equal, and $\angle$ADB need not be a special angle) ✓

$\therefore \angle \text{ADB} = \cos^{-1}\left(\frac{7}{65}\right) = 83.8°$ (1 d.p.).

Part (d) 5 marks

APPROACH

The centre P of the circle through A, B, D lies in the plane ABD and is equidistant from all three points. This gives three linear equations: $|PA|^2 = |PB|^2$, $|PA|^2 = |PD|^2$, and P lying in the plane of A, B, D.

Let $P = (x, y, z)$.

Equating $|PA|^2 = |PB|^2$,

$$(x+1)^2 + (y-2)^2 + (z-5)^2 = (x-1)^2 + (y+2)^2 + (z-8)^2$$

Expanding (the x^2, y^2, z^2 terms cancel),

$$\begin{aligned}2x - 4y - 10z + 30 &= -2x + 4y - 16z + 69 \\ \Rightarrow 4x - 8y + 6z &= 39 \quad ...(1)\end{aligned}$$

Equating $|PA|^2 = |PD|^2$, using $D = (1, 2, 8.5)$ (the y-terms cancel since A and D share $y = 2$),

$$\begin{aligned}(x+1)^2 + (z-5)^2 &= (x-1)^2 + (z-8.5)^2 \\ \Rightarrow 2x - 10z + 26 &= -2x - 17z + 73.25 \\ \Rightarrow 4x + 7z &= 47.25 \quad ...(2)\end{aligned}$$

For the third equation, P lies in the plane of A, B, D, whose normal is $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AD}$, where $\overrightarrow{AD} = D - A = \begin{pmatrix}2\\0\\3.5\end{pmatrix}$.

$$\begin{aligned}\mathbf{n} &= \begin{pmatrix}2\\-4\\3\end{pmatrix} \times \begin{pmatrix}2\\0\\3.5\end{pmatrix} \\ &= \begin{pmatrix}(-4)(3.5) - (3)(0)\\(3)(2) - (2)(3.5)\\(2)(0) - (-4)(2)\end{pmatrix} \\ &= \begin{pmatrix}-14\\-1\\8\end{pmatrix}\end{aligned}$$

Using point A, the plane equation $\mathbf{n} \cdot (P - A) = 0$ is

$$-14(x+1) - (y-2) + 8(z-5) = 0$$
$$\Rightarrow 14x + y - 8z + 52 = 0 \quad ...(3)$$

To check your answer: substituting $B = (1, -2, 8)$ into (3): $14(1) - 2 - 8(8) + 52 = 0$ ✓; substituting $D = (1, 2, 8.5)$: $14(1) + 2 - 8(8.5) + 52 = 0$ ✓

Solving (1), (2), (3) simultaneously: from (2), $x = \frac{47.25 - 7z}{4}$; from (3), $y = -52 - 14x + 8z$. Substituting both into (1),

$$4x - 8(-52 - 14x + 8z) + 6z = 39$$
$$\Rightarrow 116x - 58z = -377$$

Substituting $x = \frac{47.25 - 7z}{4}$,

$$29(47.25 - 7z) - 58z = -377$$
$$\Rightarrow -261z = -1747.25$$
$$\Rightarrow z = \frac{1747.25}{261} = \frac{241}{36}$$

Back-substituting into (2), $x = \frac{7}{72}$, and into (3), $y = \frac{7}{36}$.

To check your answer: with $P = \left(\frac{7}{72}, \frac{7}{36}, \frac{241}{36}\right)$, computing $|PA|$, $|PB|$, $|PD|$ all give exactly $\frac{65}{24}$ (the circumradius), confirming P is equidistant from A, B, D ✓

$\therefore P = \left(\frac{7}{72}, \frac{7}{36}, \frac{241}{36}\right)$.

Question 4 13 marks

Integration · Area under curve

Part (a) 3 marks

Curve C: $x = 2t^2 + 3$, $y = 5t - 1$, $t \geq \frac{1}{5}$.

APPROACH

Before setting up the integral, we check where the curve starts. At $t = \frac{1}{5}$, $y = 0$, so C begins on the x-axis, and the required area is simply the area under C from there to $x = 21$.

At $t = \frac{1}{5}$: $x = 2\left(\frac{1}{25}\right) + 3 = \frac{77}{25}$, $y = 5\left(\frac{1}{5}\right) - 1 = 0$.

At $x = 21$: $2t^2 + 3 = 21 \Rightarrow t^2 = 9 \Rightarrow t = 3$ (since $t \geq \frac{1}{5}$).

Since the curve begins on the x-axis, the required area is

$$\text{Area} = \int_{x=77/25}^{21} y \, dx = \int_{1/5}^{3} y \frac{dx}{dt} \, dt$$

With $\frac{dx}{dt} = 4t$,

$$\begin{aligned}\text{Area} &= \int_{1/5}^{3} (5t-1)(4t)\,dt \\ &= \int_{1/5}^{3} \left(20t^2 - 4t\right)\,dt \\ &= \left[\frac{20}{3}t^3 - 2t^2\right]_{1/5}^{3}\end{aligned}$$

At $t=3$: $\frac{20}{3}(27) - 2(9) = 180 - 18 = 162$. At $t=\frac{1}{5}$: $\frac{20}{3}\left(\frac{1}{125}\right) - 2\left(\frac{1}{25}\right) = -\frac{2}{75}$.

$$\text{Area} = 162 - \left(-\frac{2}{75}\right) = 162 + \frac{2}{75} = \frac{12\,152}{75}$$

To check your answer: $\frac{12\,152}{75} \approx 162.03$; since the region spans x from about 3.08 to 21 (width ≈ 17.9) with y reaching up to 14, an area around 162 is a reasonable order of magnitude ✓

$\therefore$ Area $= \frac{12\,152}{75}$.

Part (b) 5 marks

Curve D: $x = 5u$, $y = \frac{4}{u}$, $u \neq 0$.

APPROACH

At a point of intersection both parametrisations give the same (x, y). We eliminate u using the x-equations, substitute into the y-equations, and solve for t.

From C: $x = 2t^2 + 3$. From D: $x = 5u \Rightarrow u = \frac{2t^2+3}{5}$. Equating the y-values,

$$\begin{aligned}5t - 1 = \frac{4}{u} &= \frac{20}{2t^2+3} \\ \Rightarrow (5t-1)\left(2t^2+3\right) &= 20 \\ \Rightarrow 10t^3 - 2t^2 + 15t - 23 &= 0\end{aligned}$$

Testing $t=1$: $10 - 2 + 15 - 23 = 0$ ✓, so $t=1$ is a root. Dividing out the factor $(t-1)$,

$$10t^3 - 2t^2 + 15t - 23 = (t-1)\left(10t^2 + 8t + 23\right)$$

To check your answer: expanding $(t-1)\left(10t^2+8t+23\right) = 10t^3 - 2t^2 + 15t - 23$, matching the original cubic ✓

For the quadratic factor $10t^2 + 8t + 23 = 0$, the discriminant is

$$8^2 - 4(10)(23) = 64 - 920 = -856 < 0$$

Since the discriminant is negative, $10t^2 + 8t + 23 = 0$ has no real roots, so $t=1$ is the only real solution and there is only one point of intersection.

At $t=1$: $x = 2(1)^2 + 3 = 5$, $y = 5(1) - 1 = 4$.

To check your answer: at $t=1$, $u = \frac{2(1)+3}{5} = 1$; on D this gives $x = 5(1) = 5$, $y = \frac{4}{1} = 4$, matching $A = (5, 4)$ from C ✓

$\therefore A = (5, 4)$, and this is the only point of intersection.

Part (c) 5 marks

For curve C, $\frac{dy}{dt} = 5$ and $\frac{dx}{dt} = 4t$, so $\frac{dy}{dx} = \frac{5}{4t}$. At A $(t = 1)$, $\frac{dy}{dx} = \frac{5}{4}$.

The tangent to C at $A(5, 4)$ is

$$y - 4 = \frac{5}{4}(x - 5)$$

Substituting the parametrisation of D, $x = 5u,\ y = \frac{4}{u}$,

$$\frac{4}{u} - 4 = \frac{5}{4}(5u - 5)$$
$$= \frac{25}{4}(u - 1)$$

Multiplying through by $4u$,

$$16 - 16u = 25u(u - 1)$$
$$\Rightarrow 16 - 16u = 25u^2 - 25u$$
$$\Rightarrow 25u^2 - 9u - 16 = 0$$

By the quadratic formula (discriminant $= (-9)^2 - 4(25)(-16) = 1681 = 41^2$),

$$u = \frac{9 \pm 41}{50}$$
$$= 1 \text{ or } -\frac{16}{25}$$

To check your answer: $u = 1$ corresponds to A itself (the tangent touches C at A, and A also lies on D, so $u = 1$ is always a solution here); the "second time" the tangent meets D is therefore at the other root, $u = -\frac{16}{25}$ ✓

At $u = -\frac{16}{25}$,

$$x = 5u = -\frac{16}{5}, \qquad y = \frac{4}{u} = -\frac{25}{4}$$

$\therefore$ the tangent meets D a second time at $\left(-\frac{16}{5},\ -\frac{25}{4}\right)$.

Question 5 5 marks

Probability · Basic Probability Concepts

Part (a)(i) 1 mark

With 36 equally likely slots,

$$A = \{1, 3, 5, \ldots, 35\}\ (18 \text{ outcomes}), \quad B = \{2, 4, 6, \ldots, 36\}\ (18 \text{ outcomes})$$
$$C = \{3, 6, 9, \ldots, 36\}\ (12 \text{ outcomes}), \quad D = \{6, 12, 18, \ldots, 36\}\ (6 \text{ outcomes})$$

Since every score is either odd or even (never both), $A \cap B = \varnothing$. Since every multiple of 6 is even, $D \subseteq B$, so no multiple of 6 is odd and $A \cap D = \varnothing$.

Checking the remaining pairs: $A \cap C = \{3, 9, 15, 21, 27, 33\} \neq \varnothing$; $B \cap C = \{6, 12, \ldots, 36\} \neq \varnothing$; $B \cap D = D \neq \varnothing$; $C \cap D = D \neq \varnothing$.

To check your answer: A and D being mutually exclusive is consistent with $D \subseteq B$ and $A \cap B = \varnothing$ already, since $A \cap D \subseteq A \cap B = \varnothing$ ✓

$\therefore$ A and B are mutually exclusive, and A and D are mutually exclusive. No other pair is.

Part (a)(ii) 2 marks

$$P(A) = \frac{18}{36} = \frac{1}{2}, \qquad P(C) = \frac{12}{36} = \frac{1}{3}$$

$A \cap C$ is the event that the score is an odd multiple of 3: $\{3, 9, 15, 21, 27, 33\}$, so $P(A \cap C) = \frac{6}{36} = \frac{1}{6}$.

$$P(A) \times P(C) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6} = P(A \cap C)$$

Since $P(A) \times P(C) = P(A \cap C)$, A and C are independent.

$\therefore$ A and C are independent.

Checking B and C: $P(B) = \frac{1}{2}$, and $B \cap C = \{6, 12, 18, 24, 30, 36\}$, so $P(B \cap C) = \frac{6}{36} = \frac{1}{6}$.

$$P(B) \times P(C) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6} = P(B \cap C)$$

$\therefore$ B and C are another pair of independent events.

Part (b)(i) 1 mark

Now with only 35 slots (since 36 is stuck),

$A = \{1, 3, \ldots, 35\}$ (18), $\quad B = \{2, 4, \ldots, 34\}$ (17), $\quad C = \{3, 6, \ldots, 33\}$ (11), $\quad D = \{6, 12, \ldots, 30\}$ (5)

Removing slot 36 does not change the odd/even split, so $A \cap B = \varnothing$ still; and every multiple of 6 remaining (6 to 30) is still even, so $D \subseteq B$ still and $A \cap D = \varnothing$ still. The other pairs remain non-disjoint — for instance $A \cap C = \{3, 9, 15, 21, 27, 33\} \neq \varnothing$, unchanged since 36 was even and was not in $A \cap C$ to begin with.

$\therefore$ A and B, and A and D, are still the only mutually exclusive pairs — the same pairs as in (a)(i).

Part (b)(ii) 1 mark

$$P(A) = \frac{18}{35}, \qquad P(C) = \frac{11}{35}$$

$A \cap C$ is still the set of odd multiples of 3 up to 35: $\{3, 9, 15, 21, 27, 33\}$ (removing slot 36, which is even, does not affect this set), so $P(A \cap C) = \frac{6}{35}$.

$$P(A) \times P(C) = \frac{18}{35} \times \frac{11}{35} = \frac{198}{1225}$$

$$P(A \cap C) = \frac{6}{35} = \frac{210}{1225}$$

To check your answer: removing slot 36 (a multiple of both 2 and 3, but not of $A \cap C$) shrinks $P(C)$ without changing $P(A \cap C)$, so the two quantities, equal before, no longer match — exactly what breaking independence looks like ✓

Since $\frac{198}{1225} \neq \frac{210}{1225}$, $P(A) \times P(C) \neq P(A \cap C)$.

$\therefore$ A and C are no longer independent.

Question 6 8 marks

Probability · Basic Probability Concepts

Part (a) 8 marks

Let X be the number of red counters among Mei's 12 counters. Since the counters are drawn without replacement from r red and b blue counters ($r+b$ in total), X follows a hypergeometric distribution:

$$P(X=k) = \frac{\binom{r}{k}\binom{b}{12-k}}{\binom{r+b}{12}}$$

Using $P(X=4) = P(X=3)$,

$$\binom{r}{4}\binom{b}{8} = \binom{r}{3}\binom{b}{9}$$

$$\Rightarrow \frac{\binom{r}{4}}{\binom{r}{3}} = \frac{\binom{b}{9}}{\binom{b}{8}}$$

Using $\binom{r}{4} = \binom{r}{3} \cdot \frac{r-3}{4}$ and $\binom{b}{9} = \binom{b}{8} \cdot \frac{b-8}{9}$,

$$\frac{r-3}{4} = \frac{b-8}{9}$$

$$\Rightarrow 9(r-3) = 4(b-8)$$

$$\Rightarrow 9r - 27 = 4b - 32$$

$$\Rightarrow 9r + 5 = 4b$$

$\therefore$ $9r + 5 = 4b$ (shown)

Part (b)

APPROACH

We relate $P(X=3)$ to $P(X=2)$ in exactly the same way as in part (a), giving a second linear equation in r and b. Solving this simultaneously with $9r+5=4b$ then fixes r and b, from which $P(X=1)$ follows directly.

Using $P(X=3) = \frac{5}{3}P(X=2)$,

$$\binom{r}{3}\binom{b}{9} = \frac{5}{3}\binom{r}{2}\binom{b}{10}$$

Using $\binom{r}{3} = \binom{r}{2} \cdot \frac{r-2}{3}$ and $\binom{b}{10} = \binom{b}{9} \cdot \frac{b-9}{10}$, so that $\frac{\binom{b}{9}}{\binom{b}{10}} = \frac{10}{b-9}$,

$$\frac{r-2}{3} \times \frac{10}{b-9} = \frac{5}{3}$$
$$\Rightarrow 10(r-2) = 5(b-9)$$
$$\Rightarrow 2(r-2) = b-9$$
$$\Rightarrow b = 2r+5$$

Solving simultaneously with $9r + 5 = 4b$ from part (a):

$$9r + 5 = 4(2r+5)$$
$$\Rightarrow 9r + 5 = 8r + 20$$
$$\Rightarrow r = 15$$
$$b = 2(15) + 5$$
$$= 35$$

To check your answer: with $r = 15$, $b = 35$ (50 counters in total), direct computation gives $P(X = 4) = P(X = 3) \approx 0.265$ and $P(X = 3)/P(X = 2) = 5/3$ exactly ✓

Both values satisfy $r > 12$ and $b > 12$, as required by the question.

We now find the probability that exactly 1 of the 12 counters removed is red:

$$P(X = 1) = \frac{\binom{15}{1}\binom{35}{11}}{\binom{50}{12}}$$
$$= \frac{687\,735}{13\,340\,621}$$
$$\approx 0.0516 \ \ (3 \text{ s.f.})$$

$\therefore\ 2r + 5 = b$, and $r = 15$, $b = 35$; $P(X = 1) = \frac{687\,735}{13\,340\,621} \approx 0.0516$.

Question 7 7 marks

Correlation and Regression · Regression Lines

Part (a)(i) 1 mark

$$r = 0.9281 \ \ (4 \text{ d.p.})$$

Explanation: By GC, entering the eight pairs (x, y) into linear regression statistics gives the product moment correlation coefficient between x and y.

Part (a)(ii) 1 mark

$$r = 0.9697 \ \ (4 \text{ d.p.})$$

Explanation: By GC, entering the pairs (x, e^y) — where each e^y is computed from the table, e.g. $e^{3.99} \approx 54.05$ and $e^{8.39} \approx 4402.82$ — gives the product moment correlation coefficient between x and e^y.

Part (b) 3 marks

Since $r = 0.9697$ (between x and e^y) is closer to 1 than $r = 0.9281$ (between x and y), the model $e^y = cx + d$ gives a better (stronger linear) fit to the data than $y = ax + b$.

By GC, the regression line of e^y on x has

$$c \approx 375.615, \quad d \approx -1881.48$$

Note: $\bar{x} = 11$, the midpoint of the evenly spaced values $x = 4, 6, \ldots, 18$, as expected.

$\therefore\ e^y = 375.6x - 1881$ (coefficients to 4 s.f.)

Part (c) 2 marks

APPROACH

We substitute $x = 24$ into the regression equation of e^y on x from part (b), then take $\ln$ to recover y, and finally comment on reliability by checking whether $x = 24$ lies inside the data range used to fit the line.

For the year 2024, $x = 2024 - 2000 = 24$.

$$\begin{aligned} e^y &= 375.615(24) - 1881.48 \\ &\approx 7133.28 \\ y &= \ln(7133.28) \\ &\approx 8.87 \end{aligned}$$

To check your answer: 8.87 million is only a modest rise from the last data point (8.39 million in 2018), consistent with the flattening trend visible for large x ✓

Since $x = 24$ lies outside the range of the given data ($x = 4$ to $x = 18$), using the regression equation here means extrapolating beyond the data range. There is no guarantee the same relationship continues to hold there, so the estimate is not necessarily reliable.

$\therefore$ the estimated number of subscriptions in 2024 is approximately 8.87 million, but this estimate is not necessarily reliable since it requires extrapolation.

Question 8 11 marks

Hypothesis Testing · Conducting a hypothesis test

Part (a)(i) 2 marks

Let μ be the population mean mass, in kg, of granulated sugar in bags filled by the new machine.

$$H_0 : \mu = 1$$
$$H_1 : \mu \neq 1$$

This is a two-tailed test, since the manager wants to check whether the machine is correctly adjusted, i.e. whether the mean has shifted away from 1 kg in either direction.

Part (a)(ii) 2 marks

Reason 1: the sample size is only 10, too small for the Central Limit Theorem to guarantee that $\overline{X}$ is approximately normal, unless the mass of sugar per bag is already known to be normally distributed.

Reason 2: the sample is not random — always taking the first bag produced each morning and afternoon may not be representative of typical production, e.g. if the machine's fill amount drifts systematically over the course of a run.

Part (b)(i) 5 marks

> **APPROACH**
> With $n = 40$ large, the Central Limit Theorem lets us treat $\overline{X}$ as approximately normal, using the sample variance s^2 in place of the (unknown) population variance. We find the z boundary for the lower 2.5% tail, then convert it into a boundary on $\overline{x}$.

Let μ be the population mean mass, in kg, of sugar in the larger bags.

$$H_0 : \mu = 2$$
$$H_1 : \mu < 2$$

This is a one-tailed test, since the concern is specifically that the mean may be less than 2 kg.

Since $n = 40$, we first find the unbiased estimate of the population variance:

$$\begin{aligned}\bar{x} &= \frac{\sum x}{n} \\ &= \frac{78.88}{40} \\ &= 1.972\end{aligned}$$

$$\begin{aligned}s^2 &= \frac{1}{n-1}\left(\sum x^2 - \frac{(\sum x)^2}{n}\right) \\ &= \frac{1}{39}\left(155.6746 - \frac{78.88^2}{40}\right) \\ &= 0.003159 \ \ (4 \text{ s.f.})\end{aligned}$$

To check your answer: $\bar{x} = 1.972$ kg is indeed slightly below the target of 2 kg, consistent with the customers' suspicion motivating the test ✓

Since $n = 40$ is large, by the Central Limit Theorem $\overline{X} \sim N\left(2, \frac{s^2}{n}\right)$ approximately under H_0. The standard error is

$$\begin{aligned}\frac{s}{\sqrt{n}} &= \sqrt{\frac{0.003159}{40}} \\ &\approx 0.008888\end{aligned}$$

By GC, at the 2.5% significance level (one-tailed, lower tail), the critical value of Z is

$$z_{crit} = -1.9600 \ \ (4 \text{ d.p.})$$

The critical region is $\bar{x} < 2 + z_{crit} \times \frac{s}{\sqrt{n}}$:

$$\begin{aligned} \bar{x} &< 2 - 1.9600(0.008888) \\ &\approx 1.9826 \end{aligned}$$

> $\therefore$ the critical region is $\bar{x} < 1.9826 \ \ (4 \text{ d.p.})$.

Part (b)(ii) 2 marks

The observed sample mean is $\bar{x} = 1.972$, which lies in the critical region since $1.972 < 1.9826$. Equivalently, the test statistic

$$\begin{aligned} z &= \frac{1.972 - 2}{0.008888} \\ &\approx -3.15 \end{aligned}$$

is far below $z_{crit} = -1.96$.

> **To check your answer:** the test statistic and the boundary comparison agree — both place $\bar{x}$ well inside the rejection region, not just barely across it ✓

Since $\bar{x}$ falls in the critical region, there is sufficient evidence, at the 2.5% level of significance, to reject H_0.

> $\therefore$ there is sufficient evidence, at the 2.5% level, that the mean mass of sugar in the (larger) bags is less than 2 kg, supporting the customers' suspicion.

Question 9 14 marks

Normal Distribution · Scenarios involving normal distribution

Part (a) 1 mark

Let X be the length, in metres, of a randomly chosen Long plank: $X \sim N\left(1.82,\ 0.2^2\right)$.

$$P(X < 1.79) \approx 0.440 \ \ (3 \text{ s.f.})$$

Explanation: standardising,

$$\begin{aligned} P(X < 1.79) &= P\left(Z < \frac{1.79 - 1.82}{0.2}\right) \\ &= P(Z < -0.15) \\ &\approx 0.440 \ (\text{by GC}) \end{aligned}$$

Part (b) 3 marks

Let $S = X_1 + X_2 + \cdots + X_8$ be the total length of 8 randomly chosen Long planks. Since each $X_i \sim N\left(1.82,\ 0.2^2\right)$ independently,

$$S \sim N\left(8(1.82),\ 8\left(0.2^2\right)\right)$$

$$= N(14.56,\ 0.32)$$

$$P(S > 14.5) = P\left(Z > \frac{14.5 - 14.56}{\sqrt{0.32}}\right)$$

$$= P(Z > -0.106)$$

$$\approx 0.542 \ \text{(3 s.f.)}$$

To check your answer: the mean total 14.56 is only just above 14.5, so the probability of exceeding 14.5 should be just over 0.5 ✓

$\therefore\ P(S > 14.5) \approx 0.542.$

Part (c) 2 marks

Let Y be the length, in metres, of a randomly chosen Regular plank: $Y \sim N\left(1.22,\ 0.3^2\right)$.

$$P(Y > 1.25) = P\left(Z > \frac{1.25 - 1.22}{0.3}\right)$$

$$= P\left(Z > 0.1\overline{6}\right)$$

$$\approx 0.460 \ \text{(3 s.f.)}$$

Let W be the number of Sylvio's 120 planks that are longer than 1.25 m. Since each plank independently has probability ≈ 0.4602 of being longer than 1.25 m, $W \sim B(120,\ 0.4602)$ approximately, so

$$E(W) = 120 \times 0.4602$$

$$\approx 55.2$$

$\therefore\ E(W) \approx 55.2$, i.e. about 55 planks.

Part (d) 3 marks

APPROACH

We combine the two independent total-length variables as a single normal variable $D = L - R$, then use the symmetric interval $P(-0.65 < D < 0.65)$ — this interval is not centred at $E(D)$, so we standardise both endpoints separately.

Let L = total length of 10 randomly chosen Long planks and R = total length of 16 randomly chosen Regular planks.

$$L \sim N\left(10(1.82),\ 10\left(0.2^2\right)\right)$$

$$= N(18.2,\ 0.4)$$

$$R \sim N\left(16(1.22),\ 16\left(0.3^2\right)\right)$$

$$= N(19.52,\ 1.44)$$

Let $D = L - R$, the difference between the two total lengths. Since L and R are independent,

$$D \sim N(18.2 - 19.52,\ 0.4 + 1.44)$$

$$= N(-1.32,\ 1.84)$$

We require $P(|D| < 0.65) = P(-0.65 < D < 0.65)$:

$$P(-0.65 < D < 0.65) = P\left(\frac{-0.65-(-1.32)}{\sqrt{1.84}} < Z < \frac{0.65-(-1.32)}{\sqrt{1.84}}\right)$$
$$\Rightarrow P(0.494 < Z < 1.45)$$
$$\approx 0.237 \text{ (3 s.f.)}$$

To check your answer: $E(D) = -1.32$ lies outside $(-0.65,\ 0.65)$, so the interval is not centred on the mean, and a probability well below 0.5 is expected ✓

$\therefore\ P(|D| < 0.65) \approx 0.237.$

Part (e) 5 marks

APPROACH

A Short plank cut from a Long plank has length $\frac{X}{3}$; one cut from a Regular plank has length $\frac{Y}{2}$. We find the distribution of each, then of their difference $D_2 = \frac{X}{3} - \frac{Y}{2}$, and evaluate $P(D_2 > 0)$.

A Short plank cut from a Long plank has length $\frac{X}{3}$, where $X \sim N(1.82,\ 0.2^2)$:

$$\frac{X}{3} \sim N\left(\frac{1.82}{3},\ \frac{0.2^2}{9}\right)$$
$$= N(0.607,\ 0.00444) \text{ (3 s.f.)}$$

A Short plank cut from a Regular plank has length $\frac{Y}{2}$, where $Y \sim N(1.22,\ 0.3^2)$:

$$\frac{Y}{2} \sim N\left(\frac{1.22}{2},\ \frac{0.3^2}{4}\right)$$
$$= N(0.61,\ 0.0225)$$

Let $D_2 = \frac{X}{3} - \frac{Y}{2}$. Since $\frac{X}{3}$ and $\frac{Y}{2}$ are independent,

$$D_2 \sim N(0.6067 - 0.61,\ 0.004444 + 0.0225)$$
$$= N(-0.00333,\ 0.0269)$$

We require $P(D_2 > 0)$:

$$P(D_2 > 0) = P\left(Z > \frac{0-(-0.00333)}{\sqrt{0.0269}}\right)$$
$$= P(Z > 0.0203)$$
$$\approx 0.492 \text{ (3 s.f.)}$$

To check your answer: $E(D_2) \approx -0.0033$ is very close to 0, so the probability should be very close to, but just under, 0.5 ✓

$\therefore\ P\left(\frac{X}{3} > \frac{Y}{2}\right) \approx 0.492.$

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Part (f)

The Long plank needs two cuts to produce three equal Short pieces, while the Regular plank needs only one cut to produce two equal pieces. If each cut wastes a small amount of length, the Short plank from a Long plank loses length from two cuts, while the Short plank from a Regular plank loses length from only one — reducing $E\left(\frac{X}{3}\right)$ by more than $E\left(\frac{Y}{2}\right)$.

$\therefore$ the mean of $\frac{X}{3}$ decreases by more than the mean of $\frac{Y}{2}$, so the probability in part (e) would decrease.

Question 10 15 marks

Binomial Distribution · Scenarios involving binomial distribution

Part (a)(i) 2 marks

Assumption 1: Each ornament is faulty independently of whether any other ornament is faulty.

Assumption 2: Each ornament has the same, constant probability of being faulty.

To check your answer: a binomial model needs a fixed number of independent trials, each with the same probability of "success" (here, "faulty") — the two assumptions above supply exactly independence and constant probability. ✓

Part (a)(ii) 1 mark

Let X be the number of faulty ornaments in a day: $X \sim B(50,\ 0.04)$.

$$E(X) = np = 50(0.04) = 2$$

$$\text{Var}(X) = np(1-p) = 50(0.04)(0.96) = 1.92$$

$$E(X) - \text{Var}(X) = 2 - 1.92 = 0.08 \ \ (\text{shown})$$

To check your answer: in general $\text{mean} - \text{variance} = np - np(1-p) = np^2$; here $np^2 = 50(0.04)^2 = 0.08$, matching directly. ✓

Part (a)(iii) 1 mark

$$P(X \leq 2) = 0.677 \ \ (3\text{ s.f.})$$

Explanation: by GC, summing the $B(50,\ 0.04)$ probabilities from $X = 0$ to $X = 2$ gives $P(X \leq 2) = 0.676714\ldots \approx 0.677$.

Part (a)(iv) 3 marks

APPROACH

"No more than 2 faulty ornaments" is itself a Bernoulli-type event with the probability found in (iii), repeated independently over the 5 days of the week. This calls for a second binomial distribution built on top of the first, not a fresh calculation from the ornament counts.

Let Y be the number of days, out of 5, on which no more than 2 faulty ornaments are produced. Each day is independent with the same probability $P(X \leq 2) = 0.676714$ (unrounded value, from (iii)), so

$$Y \sim B(5,\ 0.676714)$$

We require $P(Y \geq 3)$, converting to cumulative form first:

$$P(Y \geq 3) = 1 - P(Y \leq 2)$$

By GC, $P(Y \leq 2) = 0.195219$, so

$$P(Y \geq 3) = 1 - 0.195219 = 0.805 \ \ (3\text{ s.f.})$$

To check your answer: the per-day probability 0.677 is above 0.5, and 3 out of 5 is close to the expected count $5(0.677) \approx 3.38$, so a fairly high probability is reasonable. ✓

Part (a)(v) 2 marks

APPROACH

the week's total fault count is a sum of 5 independent $B(50,\ 0.04)$ variables. Since they share the same p, this sum is itself binomial with $n = 5 \times 50$, rather than a normal approximation or a fresh distribution.

Let T be the total number of faulty ornaments produced in the 5-day week. Since the $5 \times 50 = 250$ ornaments made in the week are each independently faulty with probability 0.04,

$$T \sim B(250,\ 0.04)$$

$$P(T \leq 10) = 0.583 \ \ (3\text{ s.f.})$$

Explanation: by GC, $P(T \leq 10) = 0.583057\ldots \approx 0.583$.

To check your answer: $E(T) = 250(0.04) = 10$, so $P(T \leq 10)$ sitting just above 0.5 is consistent with 10 being the mean of T. ✓

Part (b) 6 marks

APPROACH

we cannot compare $P(\text{Ming accepts})$ and $P(\text{Lu accepts})$ by evaluating them at particular values of p — the question asks for *all* $0 < p < 1$. The way in is to write both probabilities as functions of p, subtract, and factorise the difference so that its sign is visible for every p in the interval.

Let $p = P(\text{pen is not faulty})$, so a pen is faulty with probability $1 - p$, where $0 < p < 1$.

Mr Lu's method: Let X_L be the number of faulty pens in his sample of 6, so $X_L \sim B(6,\ 1-p)$. He accepts the box if $X_L = 0$ or $X_L = 1$.

$$\begin{aligned} &P(\text{Lu accepts}) \\ &= P(X_L = 0) + P(X_L = 1) \\ &= p^6 + \binom{6}{1}(1-p)p^5 \end{aligned}$$

$$= p^6 + 6(1-p)p^5$$
$$= p^5\left[p + 6(1-p)\right]$$
$$= p^5(6 - 5p)$$

Mrs Ming's method: Let X_1 be the number of faulty pens in her first sample of 3, so $X_1 \sim B(3,\ 1-p)$.

If $X_1 = 0$ (probability p^3), she accepts immediately. If $X_1 = 1$ (probability $\binom{3}{1}(1-p)p^2 = 3(1-p)p^2$), she takes a second sample of 3, accepting only if that sample has 0 faulty pens (probability p^3). If $X_1 = 2$ or 3, she rejects.

$$P(\text{Ming accepts})$$
$$= p^3 + \left[3(1-p)p^2\right] \times p^3$$
$$= p^3 + 3(1-p)p^5$$

To check your answer: at $p = 0.9$, $P(\text{Lu accepts}) \approx 0.886$ and $P(\text{Ming accepts}) \approx 0.906$, so Mrs Ming does accept a greater proportion at this value. ✓

To compare the two methods for every p, we consider the difference $P(\text{Ming accepts}) - P(\text{Lu accepts})$.

$$P(\text{Ming accepts}) - P(\text{Lu accepts})$$
$$= p^3 + 3(1-p)p^5 - p^5(6-5p)$$
$$= p^3\left[1 + 3(1-p)p^2 - p^2(6-5p)\right]$$
$$= p^3\left[1 + 3p^2 - 3p^3 - 6p^2 + 5p^3\right]$$
$$= p^3\left(2p^3 - 3p^2 + 1\right)$$

Factorising $2p^3 - 3p^2 + 1$: substituting $p = 1$ gives $2 - 3 + 1 = 0$, so $(p-1)$ is a factor.

$$2p^3 - 3p^2 + 1 = (p-1)\left(2p^2 - p - 1\right)$$

The quadratic factor $2p^2 - p - 1$ has discriminant $1 + 8 = 9$, giving roots $\frac{1 \pm 3}{4} = 1$ or $-\frac{1}{2}$, so

$$2p^2 - p - 1 = (p-1)(2p+1)$$

To check your answer: expanding $(p-1)(2p+1) = 2p^2 + p - 2p - 1 = 2p^2 - p - 1$, matching. ✓

Combining the two factorisations,

$$2p^3 - 3p^2 + 1 = (p-1)(p-1)(2p+1) = (p-1)^2(2p+1)$$

so

$$P(\text{Ming accepts}) - P(\text{Lu accepts}) = p^3(p-1)^2(2p+1)$$

For $0 < p < 1$: $p^3 > 0$ since $p > 0$; $(p-1)^2 > 0$ since $p \neq 1$ on this open interval; and $2p + 1 > 0$ since $p > 0$. Hence the product $p^3(p-1)^2(2p+1) > 0$ for all $0 < p < 1$.

To check your answer: at $p = 0.5$, $p^3(p-1)^2(2p+1) = (0.125)(0.25)(2) = 0.0625$, matching the direct computation $0.171875 - 0.109375 = 0.0625$. ✓

$\therefore P(\text{Ming accepts}) - P(\text{Lu accepts}) > 0$ for all $0 < p < 1$, so Mrs Ming always accepts a strictly greater proportion of boxes than Mr Lu does, regardless of the value of p.