TEN-YEAR SERIES · WORKED SOLUTIONS

2023 A-Level H2 Mathematics Paper 1

11 questions · 100 marks

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Question 1 5 marks

Differentiation · Applications of Differentiation

Differentiating implicitly with respect to x, using the chain rule on the right-hand side (with $u = 11 - 5x$, so $\frac{du}{dx} = -5$):

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} &= 2(11-5x)\cdot(-5)\\ &= -10(11-5x)\end{aligned}$$

Multiplying through by y gives a new equation for $\frac{dy}{dx}$ itself:

$$\frac{dy}{dx} = -10y(11-5x)$$

At $x = 2$: $11 - 5(2) = 1$, so $\ln y = 1^2 = 1$, giving $y = e$.

Substituting $x = 2$ and $y = e$ into the derivative:

$$\begin{aligned}\frac{dy}{dx} &= -10e(1)\\ &= -10e\end{aligned}$$

To check your answer: at $x = 2$, $\ln y = 1 \Rightarrow y = e$, and $\frac{dy}{dx} = -10y(11-5x)$ evaluated at $x = 2$ gives $-10e(1) = -10e$, consistent ✓

The tangent at $(2, e)$ has gradient $-10e$:

$$\begin{aligned}y - e &= -10e(x-2)\\ &= -10ex + 20e\end{aligned}$$

Adding e to both sides:

$$y = -10ex + 21e$$

$\therefore$ the equation of the tangent is $y = 21e - 10ex$.

Question 2 5 marks

Sequences and Series · Concepts of Series and Sequences

Part (a) 3 marks

APPROACH

With four unknown coefficients and four data points, we set up four equations and eliminate the constant term d, then the linear term c, by taking successive differences, rather than solving the 4×4 system directly.

Let $u_n = an^3 + bn^2 + cn + d$. Substituting $n = 1, 2, 3, 4$:

$$a + b + c + d = 10 \quad ...(1)$$

$$8a + 4b + 2c + d = 61 \quad ...(2)$$
$$27a + 9b + 3c + d = 206 \quad ...(3)$$
$$64a + 16b + 4c + d = 469 \quad ...(4)$$

Taking (2)-(1), (3)-(2) and (4)-(3) eliminates d:

$$7a + 3b + c = 51 \quad ...(\text{A})$$
$$19a + 5b + c = 145 \quad ...(\text{B})$$
$$37a + 7b + c = 263 \quad ...(\text{C})$$

Taking differences again eliminates c:

$$6a + b = 47 \quad ...(\text{D})$$
$$9a + b = 59 \quad ...(\text{E})$$

Subtracting (D) from (E):

$$3a = 12$$
$$\Rightarrow a = 4$$

Back-substituting: from (D), $b = 47 - 6(4) = 23$; from (A), $c = 51 - 7(4) - 3(23) = -46$; from (1), $d = 10 - 4 - 23 - (-46) = 29$.

To check your answer: $u_1 = 4 + 23 - 46 + 29 = 10$ ✓, $u_2 = 32 + 92 - 92 + 29 = 61$ ✓, $u_3 = 108 + 207 - 138 + 29 = 206$ ✓, $u_4 = 256 + 368 - 184 + 29 = 469$ ✓

$\therefore u_n = 4n^3 + 23n^2 - 46n + 29.$

Part (b) 2 marks

$$n \geq 17$$

Explanation: By GC, $u_{16} = 21\,565$ and $u_{17} = 25\,546$, so $u_{16} < 25\,000 < u_{17}$. Since u_n is a cubic with positive leading coefficient it is increasing for $n \geq 1$, so once u_n exceeds $25\,000$ at $n = 17$ it stays above $25\,000$ for every larger n. As n is a positive integer (a term index of the sequence), the range is $n \geq 17$.

Question 3 6 marks

Vectors · Scalar and Vector Products

Part (a) 3 marks

APPROACH

We let $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ and use the fact that $\mathbf{c}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, together with the given perpendicularity condition written as a dot product equal to zero.

Since $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$:

$$\mathbf{c} \cdot \mathbf{a} = 0, \quad \mathbf{c} \cdot \mathbf{b} = 0$$

Since $(\mathbf{a}\times\mathbf{b}+\mathbf{a})$ is perpendicular to $(\mathbf{a}\times\mathbf{b}+\mathbf{b})$:

$$(\mathbf{c}+\mathbf{a})\cdot(\mathbf{c}+\mathbf{b})=0$$
$$\Rightarrow \mathbf{c}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}+\mathbf{a}\cdot\mathbf{b}=0$$

Substituting $\mathbf{c}\cdot\mathbf{b}=0$ and $\mathbf{a}\cdot\mathbf{c}=0$:

$$\Rightarrow |\mathbf{c}|^2+\mathbf{a}\cdot\mathbf{b}=0$$

Rearranging, and using $\mathbf{a}\cdot\mathbf{b}=-1$:

$$|\mathbf{a}\times\mathbf{b}|^2=-\mathbf{a}\cdot\mathbf{b}$$
$$=-(-1)$$
$$=1$$

To check your answer: $|\mathbf{a}\times\mathbf{b}|^2=1\geq 0$, consistent with a magnitude squared, and the derivation used only $\mathbf{a}\times\mathbf{b}\perp\mathbf{a}$ and $\mathbf{a}\times\mathbf{b}\perp\mathbf{b}$, which hold for any vectors $\mathbf{a},\mathbf{b}$ ✓

Since $|\mathbf{a}\times\mathbf{b}|^2=1$ and $|\mathbf{a}\times\mathbf{b}|\geq 0$:

$\therefore |\mathbf{a}\times\mathbf{b}|=1$ (shown).

Part (b) 3 marks

Let θ be the angle between the directions of $\mathbf{a}$ and $\mathbf{b}$, so $0^\circ\leq\theta\leq 180^\circ$. Using the standard results $\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}|\,|\mathbf{b}|\cos\theta$ and $|\mathbf{a}\times\mathbf{b}|=|\mathbf{a}|\,|\mathbf{b}|\sin\theta$, dividing eliminates the factor $|\mathbf{a}|\,|\mathbf{b}|$, so the magnitudes of $\mathbf{a}$ and $\mathbf{b}$ are never needed:

$$\tan\theta=\frac{|\mathbf{a}\times\mathbf{b}|}{\mathbf{a}\cdot\mathbf{b}}$$
$$=\frac{1}{-1}$$
$$=-1$$

To check your answer: since $0^\circ\leq\theta\leq 180^\circ$, $\sin\theta\geq 0$ always; here $\mathbf{a}\cdot\mathbf{b}<0$ so $\cos\theta<0$, meaning θ is obtuse — consistent with the second-quadrant solution below ✓

Since $\tan\theta=-1$ with θ obtuse:

$\therefore \theta=135^\circ$.

Question 4 10 marks

Integration · Techniques of Integration

Part (a) 2 marks

Using the product-to-sum identity $\cos A\cos B=\frac{1}{2}[\cos(A-B)+\cos(A+B)]$ with $A=px$, $B=qx$:

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$$\cos(px)\cos(qx) = \frac{1}{2}\left[\cos\left((p-q)x\right) + \cos\left((p+q)x\right)\right]$$

Since $p \neq q$ and $p \neq -q$, both $p-q$ and $p+q$ are non-zero, so integrating term by term gives sine terms throughout:

$$\begin{aligned}
&\int \cos(px)\cos(qx)\,dx \\
&= \frac{1}{2}\int \left[\cos\left((p-q)x\right) + \cos\left((p+q)x\right)\right] dx \\
&= \frac{1}{2}\left[\frac{\sin\left((p-q)x\right)}{p-q} + \frac{\sin\left((p+q)x\right)}{p+q}\right] + c
\end{aligned}$$

To check your answer: differentiating, $\frac{1}{2}\left[\cos\left((p-q)x\right) + \cos\left((p+q)x\right)\right] = \cos(px)\cos(qx)$ by the identity used ✓

$\therefore \int \cos(px)\cos(qx)\,dx = \frac{\sin((p-q)x)}{2(p-q)} + \frac{\sin((p+q)x)}{2(p+q)} + c.$

Part (b) 8 marks

Integrating by parts with $u = x$ and $dv = \cos(nx)\,dx$, so $du = dx$ and $v = \frac{\sin(nx)}{n}$:

$$\begin{aligned}
\int x\cos(nx)\,dx &= x \cdot \frac{\sin(nx)}{n} - \int \frac{\sin(nx)}{n}\,dx \\
&= \frac{x\sin(nx)}{n} - \frac{1}{n}\left(-\frac{\cos(nx)}{n}\right) + c \\
&= \frac{x\sin(nx)}{n} + \frac{\cos(nx)}{n^2} + c
\end{aligned}$$

$\therefore \int x\cos(nx)\,dx = \frac{x\sin(nx)}{n} + \frac{\cos(nx)}{n^2} + c$ (shown).

Part (c)

Using the result of (b), we evaluate at the limits $x = 0$ and $x = \pi$:

$$\int_0^{\pi} x\cos(nx)\,dx = \left[\frac{x\sin(nx)}{n} + \frac{\cos(nx)}{n^2}\right]_0^{\pi}$$

Since n is a positive integer, $\sin(n\pi) = 0$ and $\cos(n\pi) = (-1)^n$, while at $x = 0$ the bracket equals $\frac{1}{n^2}$:

$$\begin{aligned}
&= \left(0 + \frac{(-1)^n}{n^2}\right) - \left(0 + \frac{1}{n^2}\right) \\
&= \frac{(-1)^n - 1}{n^2}
\end{aligned}$$

To check your answer: for $n = 1$, $\int_0^{\pi} x\cos x\,dx = [x\sin x + \cos x]_0^{\pi} = (0-1) - (0+1) = -2$, matching $\frac{(-1)^1 - 1}{1^2} = -2$ ✓

If n is even, $(-1)^n = 1$, so the integral equals 0. If n is odd, $(-1)^n = -1$, so the integral equals $-\frac{2}{n^2}$.

$\therefore \int_0^{\pi} x\cos(nx)\,dx = \frac{k}{n^2}$, with $k = 0$ for even n and $k = -2$ for odd n.

Part (d)

APPROACH

We first need to know where $\cos(2x)$ changes sign on $\left[0, \frac{\pi}{2}\right]$ in order to remove the modulus, then apply the antiderivative from (b) with $n = 2$ separately on each piece.

On $\left[0, \frac{\pi}{2}\right]$, $\cos(2x) = 0$ at $x = \frac{\pi}{4}$, with $\cos(2x) > 0$ for $x < \frac{\pi}{4}$ and $\cos(2x) < 0$ for $x > \frac{\pi}{4}$. So $|x\cos(2x)| = x\cos(2x)$ on $\left[0, \frac{\pi}{4}\right]$ and $|x\cos(2x)| = -x\cos(2x)$ on $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$, giving

$$\int_0^{\pi/2} |x\cos(2x)|\,dx = \int_0^{\pi/4} x\cos(2x)\,dx - \int_{\pi/4}^{\pi/2} x\cos(2x)\,dx$$

Using the result of (b) with $n = 2$, let $F(x) = \frac{x\sin(2x)}{2} + \frac{\cos(2x)}{4}$. Then

$$F(0) = \frac{1}{4}, \quad F\left(\frac{\pi}{4}\right) = \frac{\pi}{8}, \quad F\left(\frac{\pi}{2}\right) = -\frac{1}{4}$$

so

$$\int_0^{\pi/4} x\cos(2x)\,dx = F\left(\frac{\pi}{4}\right) - F(0) = \frac{\pi}{8} - \frac{1}{4}$$

$$\int_{\pi/4}^{\pi/2} x\cos(2x)\,dx = F\left(\frac{\pi}{2}\right) - F\left(\frac{\pi}{4}\right) = -\frac{1}{4} - \frac{\pi}{8}$$

Subtracting:

$$\begin{aligned}\int_0^{\pi/2} |x\cos(2x)|\,dx &= \left(\frac{\pi}{8} - \frac{1}{4}\right) - \left(-\frac{1}{4} - \frac{\pi}{8}\right) \\ &= \frac{\pi}{4}\end{aligned}$$

To check your answer: $\frac{\pi}{4} \approx 0.785$, matching a numerical estimate of $\int_0^{\pi/2} |x\cos(2x)|\,dx$ ✓

$\therefore \int_0^{\pi/2} |x\cos(2x)|\,dx = \frac{\pi}{4}$.

Question 5 7 marks

Sequences and Series · Sigma Notation

Part (a) 3 marks

We split each logarithm using $\ln(r-1) + \ln(r+1) - 2\ln r$, then regroup it into a difference. Let $u_r = \ln r - \ln(r+1)$.

$$\ln\left[\frac{(r-1)(r+1)}{r^2}\right]$$

$$= \ln(r-1) + \ln(r+1) - 2\ln r$$

$$= [\ln(r-1) - \ln r] - [\ln r - \ln(r+1)]$$
$$= u_{r-1} - u_r$$

Summing from $r = 2$ to n, this telescopes:

$$\sum_{r=2}^{n} \ln\left[\frac{(r-1)(r+1)}{r^2}\right]$$
$$= \sum_{r=2}^{n} (u_{r-1} - u_r)$$
$$= u_1 - u_n$$
$$= [\ln 1 - \ln 2] - [\ln n - \ln(n+1)]$$
$$= \ln(n+1) - \ln n - \ln 2$$
$$= \ln\left(\frac{n+1}{n}\right) - \ln 2 \quad \text{(shown)}$$

To check your answer: for $n = 2$, LHS $= \ln\frac{3}{4}$ and RHS $= \ln\frac{3}{2} - \ln 2 = \ln\frac{3}{4}$ ✓

Part (b) 2 marks

Letting $n \to \infty$ in the result of (a): since $\frac{n+1}{n} = 1 + \frac{1}{n} \to 1$, $\ln\left(\frac{n+1}{n}\right) \to \ln 1 = 0$, so the partial sum

$$\sum_{r=2}^{n} \ln\left[\frac{(r-1)(r+1)}{r^2}\right] \to 0 - \ln 2 = -\ln 2 \text{ as } n \to \infty.$$

Since the partial sums tend to a finite limit, the series is convergent.

$\therefore$ the series converges, with sum to infinity $-\ln 2$.

Part (c) 2 marks

APPROACH

writing S_n for the partial sum found in (a), the terms $r = 2, \ldots, 9$ cancel between S_{20} and S_9, so the required sum is simply $S_{20} - S_9$.

Using $S_n = \sum_{r=2}^{n} \ln\left[\frac{(r-1)(r+1)}{r^2}\right] = \ln\left(\frac{n+1}{n}\right) - \ln 2$ from (a),

$$\sum_{r=10}^{20} \ln\left[\frac{(r-1)(r+1)}{r^2}\right] = S_{20} - S_9$$
$$= \left[\ln\frac{21}{20} - \ln 2\right] - \left[\ln\frac{10}{9} - \ln 2\right]$$
$$= \ln\frac{21}{20} - \ln\frac{10}{9}$$
$$= \ln\left(\frac{21/20}{10/9}\right)$$
$$= \ln\left(\frac{189}{200}\right)$$

To check your answer: direct numerical summation of $\sum_{r=10}^{20} \ln\left[\frac{(r-1)(r+1)}{r^2}\right]$ gives ≈ -0.0566, matching $\ln(189/200) \approx -0.0566$ ✓

$\therefore a = 189,\ b = 200.$

Question 6 8 marks

Integration · Volume of revolution

Part (a) 2 marks

We use $\cos 2\theta = 2\cos^2\theta - 1$, i.e. $\cos^2\theta = \frac{1+\cos 2\theta}{2}$, and square it.

$$\begin{aligned}
&\cos^4\theta \\
&= \left(\cos^2\theta\right)^2 \\
&= \left(\frac{1+\cos 2\theta}{2}\right)^2 \\
&= \frac{1 + 2\cos 2\theta + \cos^2(2\theta)}{4}
\end{aligned}$$

Applying the same identity again to $\cos^2(2\theta) = \frac{1+\cos 4\theta}{2}$,

$$\begin{aligned}
&\cos^4\theta \\
&= \frac{1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}}{4} \\
&= \frac{2 + 4\cos 2\theta + 1 + \cos 4\theta}{8} \\
&= \frac{1}{8}\left(\cos 4\theta + 4\cos 2\theta + 3\right) \quad \text{(shown)}
\end{aligned}$$

To check your answer: at $\theta = 0$, LHS $= \cos^4(0) = 1$ and RHS $= \frac{1}{8}(1+4+3) = 1$ ✓

Part (b) 6 marks

APPROACH

R lies between $x = 1.5$ and $x = 3$ on $y^4 = \left(9 - x^2\right)^3$, so $y^2 = \left(9 - x^2\right)^{3/2}$ (taking $y > 0$ in the first quadrant). The disc method then leaves $\int \left(9 - x^2\right)^{3/2}\, dx$; the substitution $x = 3\sin\theta$ turns this into $\cos^4\theta$, which part (a) already integrates.

Since R lies in the first quadrant, $y > 0$, so $y^2 = \left(9 - x^2\right)^{3/2}$. By the disc method,

$$V = \pi \int_{1.5}^{3} y^2\, dx = \pi \int_{1.5}^{3} \left(9 - x^2\right)^{3/2}\, dx.$$

Using $x = 3\sin\theta$, so $dx = 3\cos\theta\, d\theta$ and $9 - x^2 = 9\cos^2\theta$,

$$\left(9 - x^2\right)^{3/2} = \left(9\cos^2\theta\right)^{3/2} = 27\cos^3\theta.$$

When $x = 1.5$: $\sin\theta = 0.5 \Rightarrow \theta = \frac{\pi}{6}$. When $x = 3$: $\sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$.

$$V = \pi \int_{\pi/6}^{\pi/2} 27\cos^3\theta \cdot 3\cos\theta \, d\theta = 81\pi \int_{\pi/6}^{\pi/2} \cos^4\theta \, d\theta$$

Using the result of (a),

$$\begin{aligned}
&\int \cos^4\theta \, d\theta \\
&= \frac{1}{8}\int (\cos 4\theta + 4\cos 2\theta + 3)\, d\theta \\
&= \frac{\sin 4\theta}{32} + \frac{\sin 2\theta}{4} + \frac{3\theta}{8} + c
\end{aligned}$$

Let $G(\theta) = \frac{\sin 4\theta}{32} + \frac{\sin 2\theta}{4} + \frac{3\theta}{8}$. At $\theta = \frac{\pi}{2}$, $\sin 4\theta = \sin 2\pi = 0$ and $\sin 2\theta = \sin \pi = 0$, so $G\left(\frac{\pi}{2}\right) = \frac{3}{8} \cdot \frac{\pi}{2} = \frac{3\pi}{16}$. At $\theta = \frac{\pi}{6}$, $\sin 4\theta = \sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}$ and $\sin 2\theta = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$, so

$$\begin{aligned}
&G\left(\frac{\pi}{6}\right) \\
&= \frac{\sqrt{3}/2}{32} + \frac{\sqrt{3}/2}{4} + \frac{3}{8}\cdot\frac{\pi}{6} \\
&= \frac{9\sqrt{3}}{64} + \frac{\pi}{16}
\end{aligned}$$

$$\int_{\pi/6}^{\pi/2} \cos^4\theta \, d\theta = G\left(\frac{\pi}{2}\right) - G\left(\frac{\pi}{6}\right) = \frac{\pi}{8} - \frac{9\sqrt{3}}{64}$$

$$V = 81\pi\left(\frac{\pi}{8} - \frac{9\sqrt{3}}{64}\right) = \frac{81\pi\left(8\pi - 9\sqrt{3}\right)}{64}$$

To check your answer: $\frac{81\pi(8\pi - 9\sqrt{3})}{64} \approx 37.9$, matching a numerical estimate of $\pi \int_{1.5}^{3} (9 - x^2)^{3/2}\, dx$ ✓

$\therefore V = \frac{81\pi(8\pi - 9\sqrt{3})}{64}$.

Question 7 10 marks

Functions · Inverse Functions

Part (a) 3 marks

APPROACH

dropping the modulus first, we write $g(x) = \frac{2x+4}{3-x}$ as $-2 + \frac{10}{3-x}$, which exposes its asymptotes and monotonicity at once; $f = |g|$ then follows by reflecting the negative parts of g above the x-axis.

Writing $2x + 4 = -2(3 - x) + 10$,

$$\begin{aligned}
g(x) &= \frac{2x+4}{3-x} \\
&= \frac{-2(3-x)+10}{3-x}
\end{aligned}$$

$$= -2 + \frac{10}{3-x}$$

As $x \to \pm\infty$, $\frac{10}{3-x} \to 0$, so $g(x) \to -2$: g has horizontal asymptote $y = -2$, so $f = |g|$ has horizontal asymptote $y = 2$. There is a vertical asymptote $x = 3$, where the denominator vanishes.

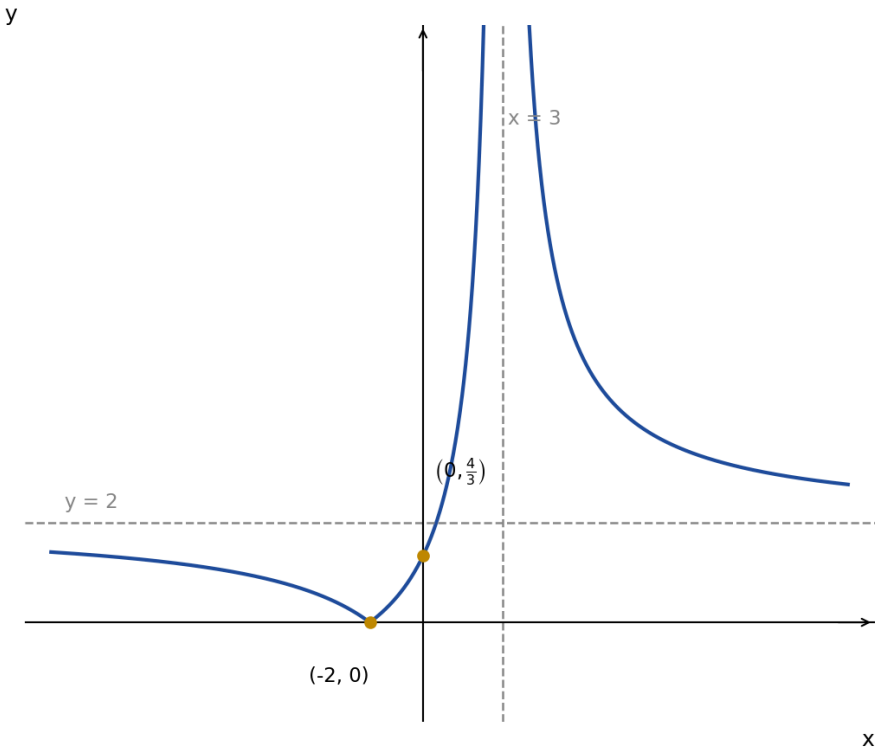
$$\frac{d}{dx}\left[-2 + \frac{10}{3-x}\right] = \frac{10}{(3-x)^2} > 0 \text{ for all } x \neq 3,$$

so g is strictly increasing on each branch. Since $g(-2) = 0$, $g(x) < 0$ for $x < -2$ and $g(x) > 0$ for $-2 < x < 3$; and $g(x) < 0$ throughout $x > 3$, since $g \to -\infty$ as $x \to 3^+$ and $g \to -2$ as $x \to \infty$.

Taking $f = |g|$: f decreases from 2 (as $x \to -\infty$) to $f(-2) = 0$, a cusp where g changes sign, then increases from 0 to $+\infty$ as $x \to 3^-$; for $x > 3$, f decreases from $+\infty$ towards 2.

x-intercept: $g(x) = 0 \Rightarrow 2x + 4 = 0 \Rightarrow x = -2$, giving $(-2, 0)$. y-intercept: $f(0) = \left|\frac{4}{3}\right| = \frac{4}{3}$, giving $\left(0, \frac{4}{3}\right)$.

To check your answer: $g(-3) = \frac{-6+4}{3+3} = -\frac{1}{3}$, so $f(-3) = \frac{1}{3}$, consistent with f decreasing towards 0 as $x \to -2^-$ ✓



∴ asymptotes $x = 3$ and $y = 2$; axis intercepts $(-2, 0)$ and $\left(0, \frac{4}{3}\right)$.

Part (b) 1 mark

From the sketch, the branch $-2 \leq x < 3$ alone rises continuously from $f(-2) = 0$ to $+\infty$, covering every value in $[0, \infty)$.

∴ range of $f = [0, \infty)$.

Part (c) 1 mark

f^2 does not exist.

Explanation: for $f \circ f$ to exist, the range of f must be a subset of the domain of f. The domain of f is all real x except $x = 3$, but the range $[0, \infty)$ includes 3 — e.g. $f(1) = \left|\frac{6}{2}\right| = 3$ — so $f\left(f(1)\right) = f(3)$ is undefined.

Part (d) 1 mark

$a = -2$

Explanation: from the sketch, f decreases on $(-\infty, -2]$ and increases on $[-2, 3)$, with a cusp (minimum) at $x = -2$. Restricting to $x \leq a$ keeps f one-one, so that f^{-1} exists, only while the domain avoids containing both a decreasing and an increasing portion; the greatest such a is where the turning point occurs.

Part (e) 4 marks

With domain restricted to $x \leq -2$, $g(x) \leq 0$ throughout, so $f(x) = -g(x) = 2 - \frac{10}{3-x}$.

Let $y = 2 - \frac{10}{3-x}$ and solve for x in terms of y.

$$\begin{aligned} y &= 2 - \frac{10}{3-x} \\ \Rightarrow \frac{10}{3-x} &= 2 - y \\ \Rightarrow 3 - x &= \frac{10}{2-y} \\ \Rightarrow x &= 3 - \frac{10}{2-y} = 3 + \frac{10}{y-2} \end{aligned}$$

Replacing y by x: $f^{-1}(x) = 3 + \frac{10}{x-2}$.

The domain of f^{-1} equals the range of the restricted f: on $x \leq -2$, $f(-2) = 0$ and $f \to 2$ (not attained) as $x \to -\infty$, so the range of the restricted f is $[0, 2)$.

To check your answer: $f^{-1}(0) = 3 + \frac{10}{0-2} = 3 - 5 = -2$, matching $f(-2) = 0$ ✓

$\therefore f^{-1}(x) = 3 + \frac{10}{x-2}$, for $0 \leq x < 2$.

Question 8 10 marks

Complex Numbers · Polar and exponential form

Part (a)(i) 2 marks

$$z = 2e^{i2\pi/3}$$

Explanation: $r = |z| = \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} = \sqrt{4} = 2$. Since $Re(z) = -1 < 0$ and $Im(z) = \sqrt{3} > 0$, z lies in the second quadrant, with reference angle $\tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$, so $\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$.

To check your answer: $2e^{i2\pi/3} = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -1 + \sqrt{3}\,i$ ✓

$\therefore z = 2e^{i2\pi/3}$.

Part (a)(ii) 3 marks

APPROACH

with z and z^* both in exponential form, $\frac{z^n}{iz^*}$ reduces to a single exponential $2^{n-1}e^{i\phi}$; "purely imaginary" then just needs $\phi \equiv \frac{\pi}{2} \pmod{\pi}$.

Since $z = 2e^{i2\pi/3}$, $z^* = 2e^{-i2\pi/3}$, and $i = e^{i\pi/2}$: $z^n = 2^n e^{i2n\pi/3}$, and

$$\begin{aligned} iz^* &= e^{i\pi/2} \cdot 2e^{-i2\pi/3} \\ &= 2e^{i(\pi/2 - 2\pi/3)} \\ &= 2e^{-i\pi/6} \end{aligned}$$

$$\begin{aligned} \frac{z^n}{iz^*} &= \frac{2^n e^{i2n\pi/3}}{2e^{-i\pi/6}} \\ &= 2^{n-1}e^{i(2n\pi/3 + \pi/6)} \end{aligned}$$

For this to be purely imaginary, its argument must satisfy $\frac{2n\pi}{3} + \frac{\pi}{6} \equiv \frac{\pi}{2} \pmod{\pi}$:

$$\begin{aligned} \frac{2n\pi}{3} + \frac{\pi}{6} &= \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z} \\ \Rightarrow 4n + 1 &= 3 + 6k \\ \Rightarrow n &= \frac{1 + 3k}{2} \end{aligned}$$

For n to be a positive integer, $1 + 3k$ must be a positive even number, so k is odd. The smallest odd k giving a positive n is $k = 1$, giving $n = 2$.

To check your answer: at $n = 2$, argument $= \frac{4\pi}{3} + \frac{\pi}{6} = \frac{3\pi}{2} \equiv \frac{\pi}{2} \pmod{\pi}$ ✓, so $\frac{z^2}{iz^*}$ is purely imaginary; at $n = 1$, argument $= \frac{5\pi}{6} \not\equiv \frac{\pi}{2} \pmod{\pi}$, so $n = 1$ fails.

$\therefore$ the smallest positive integer is $n = 2$.

Part (b) 5 marks

APPROACH

$|w|$ is real, so equating the imaginary part of $2v + |w| = 1$ forces v itself to be real; substituting $v = a$ into the second equation and comparing real and imaginary parts gives c and d in terms of a, leaving one equation in a from (1).

Let $v = a + bi$ and $w = c + di$, where $a, b, c, d \in \mathbb{R}$. Since $|w|$ is real, the equation $2v + |w| = 1$ has a real right side, so its imaginary part gives $2b = 0 \Rightarrow b = 0$: $v = a$ is real. The real part gives

$$2a + |w| = 1 \Rightarrow 2a + \sqrt{c^2 + d^2} = 1 \quad ...(1)$$

Substituting $v = a$ and $w = c + di$ into $3v - iw = -3 + 4i$,

$$\begin{aligned} &3a - i(c + di) \\ &= 3a - ci - di^2 \\ &= (3a + d) - ci \end{aligned}$$

Comparing with $-3 + 4i$: $3a + d = -3$ ...(2) and $-c = 4 \Rightarrow c = -4$ ...(3).

From (2), $d = -3 - 3a$. Substituting into (1) with $c = -4$,

$$\begin{aligned} &2a + \sqrt{16 + (-3 - 3a)^2} = 1 \\ &\Rightarrow \sqrt{16 + (3 + 3a)^2} = 1 - 2a \end{aligned}$$

Squaring both sides (valid since $1 - 2a \geq 0$, i.e. $a \leq 0.5$),

$$\begin{aligned} &16 + (3 + 3a)^2 = (1 - 2a)^2 \\ &\Rightarrow 25 + 18a + 9a^2 = 1 - 4a + 4a^2 \\ &\Rightarrow 5a^2 + 22a + 24 = 0 \end{aligned}$$

To check your answer: discriminant $= 22^2 - 4(5)(24) = 4 = 2^2$, a perfect square, so a has two rational roots ✓

$$a = \frac{-22 \pm 2}{10} \Rightarrow a = -2 \text{ or } a = -\frac{12}{5}$$

Both satisfy $a \leq 0.5$, so both are valid. Using $d = -3 - 3a$: for $a = -2$, $d = 3$; for $a = -\frac{12}{5}$, $d = -3 + \frac{36}{5} = \frac{21}{5}$.

To check your answer: for $v = -2,\ w = -4 + 3i$: $2v + |w| = -4 + 5 = 1$ ✓ and $3v - iw = -6 + 4i + 3 = -3 + 4i$ ✓. For $v = -\frac{12}{5},\ w = -4 + \frac{21}{5}i$: $2v + |w| = -\frac{24}{5} + \frac{29}{5} = 1$ ✓ and $3v - iw = -\frac{36}{5} + 4i + \frac{21}{5} = -3 + 4i$ ✓

$\therefore v = -2,\ w = -4 + 3i \quad \text{or} \quad v = -\frac{12}{5},\ w = -4 + \frac{21}{5}i.$

Question 9 13 marks

Vectors · Lines

Part (a) 5 marks

Since l_1 passes through $A(-3, 1, 2)$ with direction $\begin{pmatrix} 2 \\ 1 \\ a \end{pmatrix}$,

$$l_1 : \mathbf{r} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} + s \begin{pmatrix} 2 \\ 1 \\ a \end{pmatrix}$$

Writing l_2 in vector form, letting $\frac{x+2}{3} = \frac{y-1}{2} = z - 5 = t$,

$$l_2 : \mathbf{r} = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix} + t \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

Since l_1 and l_2 cross at B, equating components:

$$-3 + 2s = -2 + 3t \quad ...(1)$$
$$1 + s = 1 + 2t \quad ...(2)$$
$$2 + as = 5 + t \quad ...(3)$$

Equations (1) and (2) involve only s and t, so we solve these first. From (2), $s = 2t$. Substituting into (1):

$$-3 + 2(2t) = -2 + 3t$$
$$\Rightarrow -3 + 4t = -2 + 3t$$
$$\Rightarrow t = 1$$

So $s = 2t = 2$.

Substituting $s = 2$, $t = 1$ into (3):

$$2 + 2a = 5 + 1$$
$$\Rightarrow 2a = 4$$
$$\Rightarrow a = 2$$

From l_2 with $t = 1$, $B = (-2 + 3,\ 1 + 2,\ 5 + 1) = (1, 3, 6)$.

To check your answer: substituting $s = 2$ into l_1 gives $x = -3 + 4 = 1$, $y = 1 + 2 = 3$ (independent of a), matching B found from l_2. ✓

$\therefore a = 2$ and $B = (1, 3, 6)$.

Part (b)(i) 3 marks

With $a = 2$, l_1 has direction $\mathbf{d}_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$. Since π_1 is perpendicular to l_2, the normal of π_1 is the direction vector of l_2, $\mathbf{n}_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$. Since π_1 contains $A(-3, 1, 2)$,

$$\mathbf{n}_1 \cdot (\mathbf{r} - A) = 0$$
$$\Rightarrow 3(x + 3) + 2(y - 1) + (z - 2) = 0$$
$$\Rightarrow \pi_1 : \ 3x + 2y + z = -5$$

The shortest (perpendicular) distance from $B(1, 3, 6)$ to π_1 is

$$d = \frac{|3(1) + 2(3) + 6 - (-5)|}{\sqrt{3^2 + 2^2 + 1^2}}$$
$$= \frac{|3 + 6 + 6 + 5|}{\sqrt{14}}$$

$$= \frac{20}{\sqrt{14}}$$
$$= \frac{10\sqrt{14}}{7}$$
$$\approx 5.35$$

∴ the shortest distance from B to π_1 is $\frac{10\sqrt{14}}{7} \approx 5.35$ (3 s.f.).

Part (b)(ii) 2 marks

Let θ be the acute angle between l_1 and π_1 (measured from the line to the plane, not to the normal):

$$\sin\theta = \frac{|\mathbf{d}_1 \cdot \mathbf{n}_1|}{|\mathbf{d}_1|\,|\mathbf{n}_1|}$$
$$\mathbf{d}_1 \cdot \mathbf{n}_1 = (2)(3) + (1)(2) + (2)(1) = 10$$
$$|\mathbf{d}_1| = \sqrt{4+1+4} = 3, \quad |\mathbf{n}_1| = \sqrt{9+4+1} = \sqrt{14}$$
$$\sin\theta = \frac{10}{3\sqrt{14}}$$
$$\Rightarrow \theta = \sin^{-1}\left(\frac{10}{3\sqrt{14}}\right)$$
$$\approx 63.0^\circ \ (1\text{ d.p.})$$

∴ the acute angle between l_1 and π_1 is 63.0°.

Part (c) 3 marks

Approach: π_2 contains l_1, so its normal $\mathbf{n}_2$ is perpendicular to $\mathbf{d}_1$; π_2 is also perpendicular to π_1, so $\mathbf{n}_2$ is perpendicular to $\mathbf{n}_1$ too. A vector perpendicular to both $\mathbf{n}_1$ and $\mathbf{d}_1$ is $\mathbf{n}_1 \times \mathbf{d}_1$.

$$\mathbf{n}_2 = \mathbf{n}_1 \times \mathbf{d}_1 = \begin{pmatrix}3\\2\\1\end{pmatrix} \times \begin{pmatrix}2\\1\\2\end{pmatrix}$$
$$= \begin{pmatrix}(2)(2)-(1)(1)\\(1)(2)-(3)(2)\\(3)(1)-(2)(2)\end{pmatrix}$$
$$= \begin{pmatrix}3\\-4\\-1\end{pmatrix}$$

To check your answer: $\mathbf{n}_2 \cdot \mathbf{n}_1 = 9 - 8 - 1 = 0$ and $\mathbf{n}_2 \cdot \mathbf{d}_1 = 6 - 4 - 2 = 0$, so $\mathbf{n}_2$ is perpendicular to both, as required. ✓

Since π_2 contains $A(-3, 1, 2)$,

$$\mathbf{n}_2 \cdot (\mathbf{r} - A) = 0$$
$$\Rightarrow 3(x+3) - 4(y-1) - (z-2) = 0$$
$$\Rightarrow \pi_2 : \ 3x - 4y - z = -15$$

$\therefore \pi_2 : 3x - 4y - z = -15.$

Question 10 13 marks

Differential Equations · Application Problems involving DEs

Part (a) 5 marks

Andrew's energy intake is C Calories per day. Since he expends 30 Calories per day for every kg of his mass M, his energy expenditure is $30M$ Calories per day. Since the rate of change of mass is proportional to (intake - expenditure),

$$\frac{dM}{dt} = k(C - 30M)$$

where k is the constant of proportionality. (shown)

Part (b) 1 mark

$$C = 3300$$

Explanation: for Andrew's mass to remain constant at 110 kg, $\frac{dM}{dt} = 0$, so from part (a), $k\,(C - 30(110)) = 0$. Since $k \neq 0$, $C = 30(110) = 3300$.

Part (c)

Approach: substitute the new intake into the differential equation from (a), separate variables, and use the initial mass at the start of the plan to fix the constant of integration.

Andrew's new energy intake is 80% of 3300, i.e. $C' = 0.8(3300) = 2640$. Substituting into the differential equation from (a) with $C = 2640$,

$$\begin{aligned}\frac{dM}{dt} &= k(2640 - 30M) \\ &= 30k\left(\frac{2640}{30} - M\right) \\ &= 30k(88 - M)\end{aligned}$$

Separating variables,

$$\begin{aligned}\int \frac{1}{88 - M}\,dM &= \int 30k\,dt \\ \Rightarrow -\ln|88 - M| &= 30kt + c_1 \\ \Rightarrow \ln|88 - M| &= -30kt - c_1 \\ \Rightarrow 88 - M &= Ae^{-30kt}\end{aligned}$$

where $A = e^{-c_1}$. Since Andrew's mass is 110 kg at the start of the plan ($t = 0$),

$$\begin{aligned}110 &= 88 - Ae^0 \\ \Rightarrow A &= -22\end{aligned}$$

So $88 - M = -22e^{-30kt}$, i.e.

$$M = 88 + 22e^{-30kt}$$

To check your answer: at $t = 0$, $M = 88 + 22 = 110$, matching Andrew's initial mass on the plan. ✓

as required. (shown)

Part (d) 4 marks

Approach: use the data point $t = 75$, $M = 100$ to find k, then solve for the (non-integer) time at which $M = 96$, and round up since the mass must fall strictly below 96 kg.

At $t = 75$, $M = 100$:

$$\begin{aligned} 100 &= 88 + 22e^{-30k(75)} \\ \Rightarrow 12 &= 22e^{-2250k} \\ \Rightarrow e^{-2250k} &= \frac{6}{11} \\ \Rightarrow k &= \frac{\ln(11/6)}{2250} \end{aligned}$$

Setting $M = 96$,

$$\begin{aligned} 96 &= 88 + 22e^{-30kt} \\ \Rightarrow e^{-30kt} &= \frac{4}{11} \\ \Rightarrow t &= \frac{\ln(11/4)}{30k} \end{aligned}$$

Substituting $k = \frac{\ln(11/6)}{2250}$, so $30k = \frac{\ln(11/6)}{75}$,

$$\begin{aligned} t &= \frac{\ln(11/4)}{\ln(11/6)/75} \\ &= \frac{75\ln(11/4)}{\ln(11/6)} \\ &\approx 125.2 \end{aligned}$$

This is the total time (from the start of the plan) for the mass to reach 96 kg. The additional days needed after day 75:

$$t - 75 \approx 50.2$$

To check your answer: by GC, $M(125.17) = 88 + 22e^{-30k(125.17)} \approx 96.0$, matching the value $t \approx 125.2$ used above. ✓

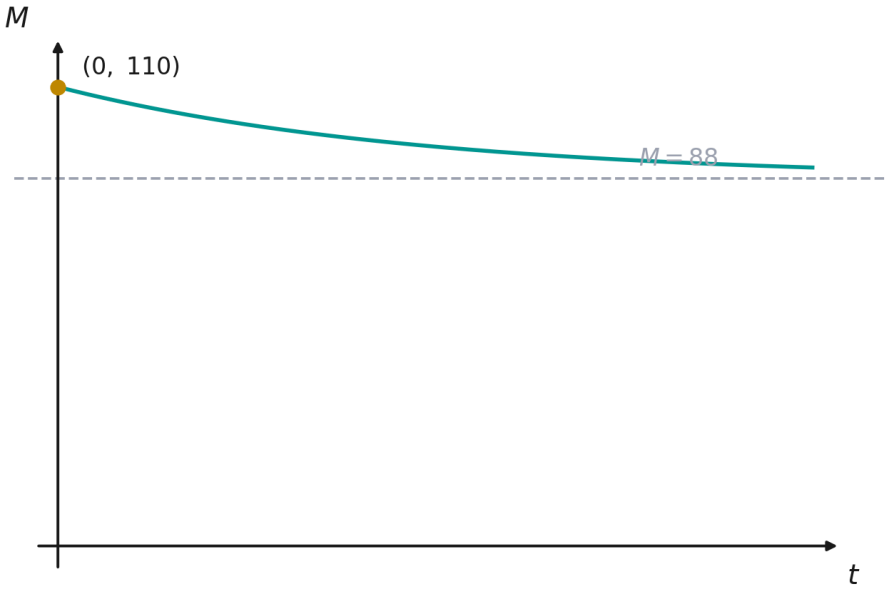
Since M decreases continuously and must fall strictly below (not just reach) 96 kg, the smallest whole number of additional days is 51.

∴ Andrew needs 51 additional days on the plan.

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Part (e)(i) 2 marks

Since $M = 88 + 22e^{-30kt}$ with $k > 0$, the term $22e^{-30kt}$ is strictly positive for every finite t, however large, so $M > 88$ for all finite t — the graph approaches the asymptote $M = 88$ but never reaches or crosses it.



∴ since $80 < 88$ and M never falls to or below 88, Andrew cannot achieve a mass of 80 kg using this plan.

Part (e)(ii) 1 mark

$$0 < C < 2400$$

Explanation: for a general intake C, the differential equation $\frac{dM}{dt} = k(C - 30M)$ has the same structure as in part (c), with equilibrium value $\frac{C}{30}$ in place of 88. Starting from $M = 110$, which is above the equilibrium whenever $C < 3300$, M decreases towards $\frac{C}{30}$ but never reaches it — exactly as in (e)(i). So Andrew's mass reaches (and passes through) 80 kg at some finite time only if the equilibrium itself lies strictly below 80:

$$\frac{C}{30} < 80 \Rightarrow C < 2400$$

Since C is an energy intake it is also positive, giving $0 < C < 2400$ Calories per day.

Question 11 13 marks

Sequences and Series · Arithmetic/Geometric Progression

Part (a) 2 marks

Approach: the deposits form an arithmetic progression; because the condition is "at least $\$50\,000$ " and a is wanted to the nearest cent, we solve the equality first and then check whether rounding up or down is what actually satisfies the inequality.

From 31 January 2021 to 31 December 2023 is exactly 3 years, i.e. 36 monthly deposits, forming an arithmetic progression with first term $\$a$, common difference $\$50$, and 36 terms.

$$\begin{aligned} S_{36} &= \frac{36}{2}\left[2a + (36-1)(50)\right] \\ &= 18(2a + 1750) \\ &= 36a + 31500 \end{aligned}$$

Requiring $S_{36} \geq 50\,000$,

$$\begin{aligned} 36a + 31500 &\geq 50\,000 \\ \Rightarrow 36a &\geq 18\,500 \\ \Rightarrow a &\geq \frac{18\,500}{36} \\ &= 513.8\overline{8} \end{aligned}$$

Since a is to the nearest cent, we must round up (rounding down would leave $S_{36} < 50\,000$).

To check your answer: at $a = \$513.89$, $S_{36} = 36(513.89) + 31500 = \$50\,000.04 \geq 50\,000$; at $a = \$513.88$, $S_{36} = \$49\,999.68 < 50\,000$ — confirming $\$513.89$ is indeed the smallest cent value that works. ✓

∴ the smallest value of a is $\$513.89$.

Part (b) 3 marks

Approach: set up a recurrence for the amount owed, unroll it for the first few months to see the pattern, then sum the resulting geometric series in general.

Let L_n be the amount owed after n months. Each month, interest of $0.1\% = 0.001$ is added first, then the repayment x is subtracted:

$$L_n = 1.001L_{n-1} - x, \quad L_0 = 400\,000$$

Applying this repeatedly,

$$\begin{aligned} L_1 &= 1.001L_0 - x \\ L_2 &= 1.001L_1 - x \\ &= (1.001)^2 L_0 - x(1.001 + 1) \\ L_3 &= 1.001L_2 - x \\ &= (1.001)^3 L_0 - x\left((1.001)^2 + 1.001 + 1\right) \end{aligned}$$

In general,

$$L_n = (1.001)^n L_0 - x\left((1.001)^{n-1} + (1.001)^{n-2} + \cdots + 1.001 + 1\right)$$

The bracketed term is a geometric series of n terms with first term 1 and common ratio 1.001:

$$\begin{aligned} &(1.001)^{n-1} + \cdots + 1 \\ &= \frac{(1.001)^n - 1}{1.001 - 1} \\ &= 1000\left((1.001)^n - 1\right) \end{aligned}$$

Substituting back,

$$L_n = 400\,000(1.001)^n - 1000x\left((1.001)^n - 1\right)$$

To check your answer: at $n = 1$, the formula gives $400\,000(1.001) - 1000x(1.001 - 1) = 400\,400 - x$, matching the recurrence $L_1 = 1.001(400\,000) - x = 400\,400 - x$ directly. ✓

as required. (shown)

Part (c)(i) 4 marks

For the loan to be fully repaid in 360 months, $L_{360} = 0$:

$$400\,000(1.001)^{360} - 1000x\left((1.001)^{360} - 1\right) = 0$$
$$\Rightarrow 1000x\left((1.001)^{360} - 1\right) = 400\,000(1.001)^{360}$$
$$x = \frac{400(1.001)^{360}}{(1.001)^{360} - 1}$$
$$\approx 1323.6348$$
$$\approx \$1323.63$$

$\therefore$ the monthly repayment is $\$1323.63$ (nearest cent).

Part (c)(ii)

Using the rounded repayment from (c)(i), the total repaid over 360 months is $360 \times \$1323.63$. The total interest paid is this total minus the original loan amount:

$$\text{Total interest} = 360(1323.63) - 400\,000$$
$$= 476\,506.80 - 400\,000$$
$$= \$76\,506.80$$

$\therefore$ the total interest paid is $\$76\,506.80$.

Part (d)(i) 4 marks

Approach: with a fixed $\$1600$ payment, use the formula from (b) to write L_n explicitly, find the (non-integer) n at which $L_n = 0$, and note that Wei cannot overpay — so she pays $\$1600$ for the largest whole number of months k that still leaves a positive balance, then clears the rest with a smaller final payment y.

With $x = 1600$ in the formula from (b),

$$L_n = 400\,000(1.001)^n - 1\,600\,000\left((1.001)^n - 1\right)$$
$$= 1\,600\,000 - 1\,200\,000(1.001)^n$$

L_n is decreasing (since $(1.001)^n$ grows), reaching 0 when

$$1\,600\,000 - 1\,200\,000(1.001)^n = 0$$
$$\Rightarrow (1.001)^n = \frac{4}{3}$$
$$\Rightarrow n = \frac{\ln(4/3)}{\ln(1.001)}$$

$$\approx 287.83$$

Since n is not a whole number, Wei makes $k = 287$ full payments of $\$1600$, leaving a small positive balance that is cleared by a smaller final payment.

Note: a 288th payment of $\$1600$ would overpay, since $287.83 < 288$.

$$L_{287} = 1\,600\,000 - 1\,200\,000(1.001)^{287}$$
$$\approx 1320.22$$

The final repayment clears this balance after one more month's interest is added:

$$y = 1.001\, L_{287}$$
$$\approx 1321.54$$

To check your answer: $y \approx 1321.54 < 1600$ as required; a further $\$1600$ payment at month 288 instead would give $1.001(1320.22) - 1600 \approx -278.46$, i.e. an overpayment, confirming $k = 287$ and not 288. ✓

$\therefore k = 287$ and $y \approx \$1321.54$.

Part (d)(ii)

The total repaid under the plan in (d)(i):

$$1600(287) + 1321.54 = 460\,521.54$$

The total repaid under the plan in (c)(i):

$$360(1323.63) = 476\,506.80$$
$$\text{Saving} = 476\,506.80 - 460\,521.54$$
$$\approx \$15\,985.26$$
$$\approx \$15\,990 \ \ (4 \text{ s.f.})$$

$\therefore$ Wei saves approximately $\$15\,990$ (4 s.f.) by making the repayments in (d)(i) rather than in (c)(i).

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