TEN-YEAR SERIES · WORKED SOLUTIONS

2022 A-Level H2 Mathematics Paper 1

12 questions · 100 marks

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Question 1 4 marks

Complex Numbers · Properties of Complex Numbers

APPROACH

With two linear equations in z and w, we eliminate one variable by substitution, then rationalise the resulting complex fraction to reach the $a+ib$ form.

$$iz+2w=-1 \quad ...(1)$$
$$(2-i)z+iw=6 \quad ...(2)$$

From (1),

$$w=\frac{-1-iz}{2}$$

Substituting into (2),

$$(2-i)z+i\left(\frac{-1-iz}{2}\right)=6$$

Multiplying through by 2,

$$2(2-i)z+i(-1-iz)=12$$
$$\Rightarrow (4-2i)z-i-i^2z=12$$

Since $-i^2z=z$,

$$(4-2i)z+z-i=12$$
$$\Rightarrow (5-2i)z=12+i$$
$$\Rightarrow z=\frac{12+i}{5-2i}$$

Multiplying numerator and denominator by the conjugate $5+2i$,

$$z=\frac{(12+i)(5+2i)}{(5-2i)(5+2i)}$$
$$=\frac{60+24i+5i+2i^2}{5^2+2^2}$$
$$=\frac{58+29i}{29}$$
$$=2+i$$

Substituting back into $w=\frac{-1-iz}{2}$,

$$w=\frac{-1-i(2+i)}{2}$$
$$=\frac{-1-2i-i^2}{2}$$
$$=\frac{-2i}{2}$$
$$=-i$$

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To check your answer: in (1), $i(2+i)+2(-i)=2i+i^2-2i=-1$ ✓; in (2), $(2-i)(2+i)+i(-i)=(4-i^2)+(-i^2)=5+1=6$ ✓

$\therefore z=2+i$ and $w=-i$.

Question 2 6 marks

Maclaurin Series · Deriving Maclaurin Series

Part (a) 3 marks

$$f(x)=\tan^{-1}\left(\sqrt{2}+x\right)$$

Using $\frac{d}{du}\tan^{-1}u=\frac{1}{1+u^2}$ with $u=\sqrt{2}+x$, where $\frac{du}{dx}=1$,

$$f'(x)=\frac{1}{1+\left(\sqrt{2}+x\right)^2}$$

Writing $f'(x)=\left[1+\left(\sqrt{2}+x\right)^2\right]^{-1}$ and differentiating by the chain rule,

$$\begin{aligned}f''(x)&=-\left[1+\left(\sqrt{2}+x\right)^2\right]^{-2}\times 2\left(\sqrt{2}+x\right)\\&=-\frac{2\left(\sqrt{2}+x\right)}{\left[1+\left(\sqrt{2}+x\right)^2\right]^2}\end{aligned}$$

To check your answer: at $x=0$, $f'(0)=\frac{1}{1+2}=\frac{1}{3}$, finite and consistent with f being smooth at $x=0$ ✓

$\therefore f'(x)=\frac{1}{1+\left(\sqrt{2}+x\right)^2}$ and $f''(x)=-\frac{2\left(\sqrt{2}+x\right)}{\left[1+\left(\sqrt{2}+x\right)^2\right]^2}$.

Part (b) 3 marks

Evaluating the derivatives from (a) at $x=0$,

$$\begin{aligned}f(0)&=\tan^{-1}\left(\sqrt{2}\right)\approx 0.9553\\f'(0)&=\frac{1}{1+2}=\frac{1}{3}\\f''(0)&=-\frac{2\sqrt{2}}{3^2}=-\frac{2\sqrt{2}}{9}\approx -0.3143\end{aligned}$$

Using the Maclaurin series $f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots$,

$$\begin{aligned}f(x)&\approx 0.9553+\frac{1}{3}x+\frac{-2\sqrt{2}/9}{2}x^2\\&=0.9553+\frac{1}{3}x-\frac{\sqrt{2}}{9}x^2\end{aligned}$$

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To check your answer: at $x = 0.1$, the series gives $0.9553 + 0.0333 - 0.00157 \approx 0.9871$; direct computation gives $\tan^{-1}(\sqrt{2} + 0.1) \approx 0.9871$, matching closely for this small x ✓

Rounding each coefficient to 3 significant figures ($\tan^{-1}\sqrt{2} \approx 0.955$, $\frac{1}{3} \approx 0.333$, $-\frac{\sqrt{2}}{9} \approx -0.157$),

$$f(x) \approx 0.955 + 0.333x - 0.157x^2$$

∴ the first three terms are $f(x) \approx 0.955 + 0.333x - 0.157x^2$.

Question 3 6 marks

Differentiation · Techniques of Differentiation

Part (a) 3 marks

$$x = \frac{1}{2}\left(e^{3t} + 2e^{-3t}\right), \quad y = \frac{1}{2}\left(e^{3t} - 2e^{-3t}\right)$$

Differentiating with respect to t,

$$\frac{dx}{dt} = \frac{1}{2}\left(3e^{3t} - 6e^{-3t}\right) = \frac{3}{2}e^{3t} - 3e^{-3t}$$
$$\frac{dy}{dt} = \frac{1}{2}\left(3e^{3t} + 6e^{-3t}\right) = \frac{3}{2}e^{3t} + 3e^{-3t}$$

At $t = \frac{1}{3}\ln 2$, $e^{3t} = e^{\ln 2} = 2$ and $e^{-3t} = \frac{1}{2}$, so

$$\frac{dx}{dt} = \frac{3}{2}(2) - 3\left(\frac{1}{2}\right) = \frac{3}{2}$$
$$\frac{dy}{dt} = \frac{3}{2}(2) + 3\left(\frac{1}{2}\right) = \frac{9}{2}$$

The gradient of the tangent is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$
$$= \frac{9/2}{3/2}$$
$$= 3$$

To check your answer: at $t = \frac{1}{3}\ln 2$, $x = \frac{1}{2}(2+1) = \frac{3}{2}$ and $y = \frac{1}{2}(2-1) = \frac{1}{2}$, both finite, and $\frac{dx}{dt} \neq 0$ there, so the curve is smooth with a well-defined tangent at this point ✓

Since the tangent has gradient 3, the normal (perpendicular to the tangent) has gradient $-\frac{1}{3}$.

∴ the gradient of the normal is $-\frac{1}{3}$.

Part (b) 3 marks

APPROACH

the parameter appears as e^{3t} and e^{-3t} in both x and y. Squaring x and y and subtracting eliminates the cross terms and the parameter together, since x^2 and y^2 each contain e^{6t} and e^{-6t} with the same coefficients but opposite signs on the middle term.

$$x^2 = \frac{1}{4}\left(e^{3t} + 2e^{-3t}\right)^2$$
$$= \frac{1}{4}\left(e^{6t} + 4 + 4e^{-6t}\right)$$
$$y^2 = \frac{1}{4}\left(e^{3t} - 2e^{-3t}\right)^2$$
$$= \frac{1}{4}\left(e^{6t} - 4 + 4e^{-6t}\right)$$

Subtracting,

$$x^2 - y^2 = \frac{1}{4}\left[\left(e^{6t} + 4 + 4e^{-6t}\right) - \left(e^{6t} - 4 + 4e^{-6t}\right)\right]$$
$$= \frac{1}{4}(8)$$
$$= 2$$

To check your answer: at $t = \frac{1}{3}\ln 2$, $x = \frac{3}{2}$, $y = \frac{1}{2}$, so $x^2 - y^2 = \frac{9}{4} - \frac{1}{4} = 2$, matching ✓

For the restriction on x: since $e^{3t} > 0$ and $2e^{-3t} > 0$ for all real t, by AM–GM,

$$e^{3t} + 2e^{-3t} \geq 2\sqrt{e^{3t} \cdot 2e^{-3t}} = 2\sqrt{2}$$

with equality when $e^{3t} = 2e^{-3t}$, i.e. $e^{6t} = 2$. Hence

$$x = \frac{1}{2}\left(e^{3t} + 2e^{-3t}\right) \geq \frac{1}{2}\left(2\sqrt{2}\right) = \sqrt{2}$$

To check your answer: setting $\frac{dx}{dt} = 0$ gives $\frac{3}{2}e^{3t} - 3e^{-3t} = 0 \Rightarrow e^{6t} = 2$, the same condition found from AM–GM, confirming $x = \sqrt{2}$ is the minimum (turning) value of x ✓

$\therefore$ the cartesian equation is $x^2 - y^2 = 2$, with $x \geq \sqrt{2}$.

Question 4 7 marks

Integration · Techniques of Integration

Part (a) 2 marks

We write $\cot x$ in terms of sine and cosine, then differentiate with the quotient rule.

$$\cot x = \frac{\cos x}{\sin x}$$
$$\frac{d}{dx}(\cot x) = \frac{(-\sin x)(\sin x) - (\cos x)(\cos x)}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$
$$= \frac{-\left(\sin^2 x + \cos^2 x\right)}{\sin^2 x}$$

Since $\sin^2 x + \cos^2 x = 1$,

$$\frac{d}{dx}(\cot x) = \frac{-1}{\sin^2 x} = -cosec^2 x \quad \text{(shown)}$$

To check your answer: at $x = \frac{\pi}{4}$, $\cot x = 1$, and numerically differentiating $\cot x$ there gives a gradient close to -2, matching $-cosec^2\left(\frac{\pi}{4}\right) = -\left(\sqrt{2}\right)^2 = -2$ ✓

Part (b) 1 mark

Using $\sin 2x = 2\sin x \cos x$ and $\tan x = \frac{\sin x}{\cos x}$,

$$\sin 2x \tan x = (2\sin x \cos x)\left(\frac{\sin x}{\cos x}\right)$$
$$= 2\sin^2 x \quad \text{(shown)}$$

To check your answer: at $x = \frac{\pi}{6}$, LHS $= \sin\frac{\pi}{3}\tan\frac{\pi}{6} = \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{3}} = \frac{1}{2}$; RHS $= 2\sin^2\frac{\pi}{6} = 2\left(\frac{1}{2}\right)^2 = \frac{1}{2}$ ✓

Part (c) 4 marks

APPROACH

the integrand is $cosec\,6x\cot 3x$, which is not a standard form. Part (b) supplies an identity relating $\sin 2x$, $\tan x$ and $\sin^2 x$; replacing x by $3x$ turns it into an identity for $\sin 6x$, $\tan 3x$ and $\sin^2 3x$, from which $cosec\,6x\cot 3x$ can be rewritten as a multiple of $cosec^2 3x$, a form we can integrate directly.

Replacing x by $3x$ in (b), $\sin 6x \tan 3x = 2\sin^2 3x$. Rearranging for $\cot 3x$,

$$\tan 3x = \frac{2\sin^2 3x}{\sin 6x}$$
$$\Rightarrow \cot 3x = \frac{\sin 6x}{2\sin^2 3x}$$

Multiplying both sides by $cosec\,6x = \frac{1}{\sin 6x}$,

$$cosec\,6x\cot 3x = \frac{1}{\sin 6x}\cdot\frac{\sin 6x}{2\sin^2 3x}$$
$$= \frac{1}{2}cosec^2 3x$$

To check your answer: at $x = \frac{\pi}{12}$, $cosec\left(\frac{\pi}{2}\right)\cot\left(\frac{\pi}{4}\right) = (1)(1) = 1$; $\frac{1}{2}\text{cosec}^2\left(\frac{\pi}{4}\right) = \frac{1}{2}\left(\sqrt{2}\right)^2 = 1$ ✓

By (a) generalized with the chain rule, $\frac{d}{dx}[\cot(3x)] = -3\,cosec^2(3x)$, so

$$\int cosec\,6x\cot 3x\,dx = \int \frac{1}{2}cosec^2 3x\,dx$$
$$= \frac{1}{2}\left(-\frac{1}{3}\cot 3x\right) + c$$

$$= -\frac{1}{6}\cot 3x + c$$

Evaluating from $x = \frac{\pi}{18}$ to $x = \frac{\pi}{9}$: at $x = \frac{\pi}{9}$, $3x = \frac{\pi}{3}$, so $\cot\left(\frac{\pi}{3}\right) = \frac{1}{\sqrt{3}}$; at $x = \frac{\pi}{18}$, $3x = \frac{\pi}{6}$, so $\cot\left(\frac{\pi}{6}\right) = \sqrt{3}$.

$$\int_{\pi/18}^{\pi/9} cosec\, 6x \cot 3x\, dx = \left[-\frac{1}{6}\cot 3x\right]_{\pi/18}^{\pi/9}$$
$$= -\frac{1}{6}\left(\frac{1}{\sqrt{3}} - \sqrt{3}\right)$$
$$= -\frac{1}{6}\left(\frac{1-3}{\sqrt{3}}\right)$$
$$= \frac{2}{6\sqrt{3}}$$
$$= \frac{1}{3\sqrt{3}}$$

Rationalizing,

$$\frac{1}{3\sqrt{3}} = \frac{\sqrt{3}}{9}$$

To check your answer: $\frac{\sqrt{3}}{9} \approx 0.1925$, matching a numerical estimate of $\int_{\pi/18}^{\pi/9} cosec\, 6x \cot 3x\, dx \approx 0.1925$ ✓

$\therefore \int_{\pi/18}^{\pi/9} cosec\, 6x \cot 3x\, dx = \frac{\sqrt{3}}{9}$.

Question 5 8 marks

Graphs and Transformations · Properties of graphs

Part (a) 4 marks

APPROACH

a line is tangent to a circle exactly when the perpendicular distance from the centre to the line equals the radius. Writing $y = mx$ as $mx - y = 0$ lets us use the standard point-to-line distance formula directly.

The circle $(x+8)^2 + (y-14)^2 = 52$ has centre $(-8, 14)$ and radius $\sqrt{52}$. The line $y = mx$ can be written $mx - y = 0$. Since it is tangent to the circle, the perpendicular distance from the centre to the line equals the radius.

$$\frac{|m(-8) - 14|}{\sqrt{m^2+1}} = \sqrt{52}$$
$$\Rightarrow \frac{|-8m - 14|}{\sqrt{m^2+1}} = \sqrt{52}$$

Squaring both sides,

$$\frac{(8m+14)^2}{m^2+1} = 52$$
$$\Rightarrow (8m+14)^2 = 52\left(m^2+1\right)$$
$$\Rightarrow 64m^2+224m+196 = 52m^2+52$$
$$\Rightarrow 12m^2+224m+144 = 0$$

Dividing through by 4,

$$3m^2+56m+36 = 0 \quad \text{(shown)}$$

To check your answer: dividing $12m^2+224m+144=0$ by 4 gives $3m^2+56m+36=0$ directly, matching the target equation ✓

Part (b) 4 marks

APPROACH

part (a) gives the gradients m of the two tangents from the origin. Each point of tangency is the foot of the perpendicular from the circle's centre to its tangent line, so once m is known for each tangent, the corresponding point can be found directly without solving the circle and line simultaneously.

Solving $3m^2+56m+36=0$ by the quadratic formula, discriminant $=56^2-4(3)(36) = 3136-432 = 2704 = 52^2$,

$$m = \frac{-56 \pm 52}{6}$$
$$\Rightarrow m = -\frac{2}{3} \text{ or } m = -18$$

To check your answer: product of roots $=\frac{36}{3}=12$, and $\left(-\frac{2}{3}\right)(-18)=12$ ✓. Sum of roots $=-\frac{56}{3}$, and $-\frac{2}{3}-18 = -\frac{2+54}{3} = -\frac{56}{3}$ ✓

The point of tangency for a tangent $y=mx$ is the foot of the perpendicular from the centre $(-8,14)$ to the line. Parametrizing the line as (t, mt), this foot is at

$$t = \frac{x_0+my_0}{1+m^2}, \quad \text{point} = (t,\ mt)$$

where $(x_0, y_0) = (-8, 14)$.

For $m=-\frac{2}{3}$ (point A),

$$t = \frac{-8+\left(-\frac{2}{3}\right)(14)}{1+\frac{4}{9}}$$
$$= \frac{-8-\frac{28}{3}}{\frac{13}{9}}$$
$$= \frac{-\frac{52}{3}}{\frac{13}{9}}$$
$$= -\frac{52}{3}\times\frac{9}{13}$$
$$= -12$$

so $y = -\frac{2}{3}(-12) = 8$, giving $A = (-12,\ 8)$.

For $m = -18$ (point B),

$$\begin{aligned} t &= \frac{-8 + (-18)(14)}{1 + 324} \\ &= \frac{-8 - 252}{325} \\ &= -\frac{260}{325} \\ &= -\frac{4}{5} \end{aligned}$$

so $y = -18\left(-\frac{4}{5}\right) = \frac{72}{5}$, giving $B = \left(-\frac{4}{5},\ \frac{72}{5}\right)$.

To check your answer: on the circle, $A = (-12, 8)$ gives $(-12+8)^2 + (8-14)^2 = 16 + 36 = 52$ ✓, and on the line, $8 = -\frac{2}{3}(-12) = 8$ ✓. For $B = \left(-\frac{4}{5}, \frac{72}{5}\right)$, $\left(-\frac{4}{5}+8\right)^2 + \left(\frac{72}{5} - 14\right)^2 = \left(\frac{36}{5}\right)^2 + \left(\frac{2}{5}\right)^2 = \frac{1296+4}{25} = \frac{1300}{25} = 52$ ✓, and on the line $-18\left(-\frac{4}{5}\right) = \frac{72}{5}$ ✓

$\therefore A = (-12,\ 8)$ and $B = \left(-\frac{4}{5},\ \frac{72}{5}\right)$.

Question 6 9 marks

Functions · Inverse Functions

Part (a) 4 marks

APPROACH

$y = \frac{1}{x}$ is transformed by translations and a stretch. To see these directly, we first rewrite $f(x) = \frac{ax+k}{x-a}$ in the form $\frac{c}{x-a} + a$ by polynomial division, which matches the shape of a transformed $\frac{1}{x}$.

Writing $ax + k = a(x - a) + \left(k + a^2\right)$, since $a(x-a) = ax - a^2$ leaves remainder $k + a^2$,

$$\begin{aligned} f(x) &= \frac{a(x-a) + \left(k + a^2\right)}{x - a} \\ &= a + \frac{k + a^2}{x - a} \end{aligned}$$

To check your answer: expanding $a + \frac{k+a^2}{x-a} = \frac{a(x-a)+k+a^2}{x-a} = \frac{ax-a^2+k+a^2}{x-a} = \frac{ax+k}{x-a}$, matching the original $f(x)$ ✓

Starting from $y = \frac{1}{x}$, the curve $y = f(x) = \frac{k+a^2}{x-a} + a$ is obtained by:

Step 1: a translation of a units in the positive x-direction (replacing x with $x - a$), giving $y = \frac{1}{x-a}$.

Step 2: a stretch parallel to the y-axis, scale factor $\left(a^2 + k\right)$, giving $y = \frac{a^2+k}{x-a}$.

Step 3: a translation of a units in the positive y-direction, giving $y = \frac{a^2+k}{x-a} + a = f(x)$.

$\therefore y = \frac{1}{x}$ is translated a units in the positive x-direction, stretched parallel to the y-axis with scale factor $a^2 + k$, then translated a units in the positive y-direction.

Part (b) 2 marks

Let $y = \frac{ax+k}{x-a}$. Solving for x in terms of y,

$$\begin{aligned} y(x-a) &= ax + k \\ \Rightarrow xy - ay &= ax + k \\ \Rightarrow xy - ax &= k + ay \\ \Rightarrow x(y-a) &= k + ay \\ \Rightarrow x &= \frac{ay+k}{y-a} \end{aligned}$$

Swapping x and y,

$$f^{-1}(x) = \frac{ax+k}{x-a}$$

To check your answer: $f^{-1}(x)$ has exactly the same formula as $f(x)$. Testing with $a = 2$, $k = 3$, $x = 5$: $f(5) = \frac{13}{3}$, and $f\left(\frac{13}{3}\right) = 5$, so f maps $5 \to \frac{13}{3} \to 5$, confirming f is self-inverse ✓

$\therefore f^{-1}(x) = \frac{ax+k}{x-a} = f(x)$.

Part (c) 1 mark

$$f^2(x) = x$$

Explanation: by (b), $f^{-1}(x) = f(x)$, so f is an involution (self-inverse). Applying f twice returns the original input, $f^2(x) = f\left(f(x)\right) = f\left(f^{-1}(x)\right) = x$.

Part (d) 2 marks

APPROACH

since $f^2(x) = x$ for every x, applying f an even number of times is the identity, and applying it an odd number of times is the same as applying it once. As 2023 is odd, f^{2023} reduces to a single application of f.

Since 2022 is even, $f^{2022}(y) = y$ for any y, so

$$\begin{aligned} f^{2023}(x) &= f^{2022}\left(f(x)\right) \\ &= f(x) \end{aligned}$$

Hence $f^{2023}(1) = f(1)$.

$$\begin{aligned} f(1) &= \frac{a(1)+k}{1-a} \\ &= \frac{a+k}{1-a} \end{aligned}$$

To check your answer: testing with $a = 2$, $k = 3$: direct iteration of f starting from 1, applied 2023 times, gives -5, matching $\frac{a+k}{1-a} = \frac{2+3}{1-2} = \frac{5}{-1} = -5$ ✓

$\therefore f^{2023}(1) = \frac{a+k}{1-a}$.

Question 7 9 marks

Integration · Area under curve

Part (a) 4 marks

We differentiate $y = x^{-3}\ln x$ with the product rule, taking $u = x^{-3}$, $v = \ln x$.

$$\begin{aligned}\frac{dy}{dx} &= -3x^{-4}\ln x + x^{-3}\cdot\frac{1}{x}\\ &= -3x^{-4}\ln x + x^{-4}\\ &= x^{-4}(1 - 3\ln x)\\ \frac{dy}{dx} &= \frac{1-3\ln x}{x^4}\quad(\text{shown})\end{aligned}$$

To check your answer: at $x = 1$, $\frac{dy}{dx} = \frac{1-0}{1} = 1$, and numerically differentiating $y = x^{-3}\ln x$ near $x = 1$ gives a gradient close to 1 ✓

At a turning point, $\frac{dy}{dx} = 0$.

$$\begin{aligned}\frac{1-3\ln x}{x^4} &= 0\\ \Rightarrow 1 - 3\ln x &= 0\\ \Rightarrow \ln x &= \frac{1}{3}\\ \Rightarrow x &= e^{1/3}\end{aligned}$$

Substituting into $y = x^{-3}\ln x$,

$$\begin{aligned}y &= \left(e^{1/3}\right)^{-3}\ln\left(e^{1/3}\right)\\ &= e^{-1}\times\frac{1}{3}\\ &= \frac{1}{3e}\end{aligned}$$

To check your answer: at $x = e^{1/3} \approx 1.396$, direct computation gives $x^{-3}\ln x \approx 0.1226$, matching $\frac{1}{3e} \approx 0.1226$ ✓

$\therefore$ the turning point is $\left(e^{1/3}, \frac{1}{3e}\right)$.

Part (b) 5 marks

APPROACH

the region is bounded by the curve, the x-axis and $x = 3$; we first need the x-intercept to know where the enclosed region starts. Since $x^{-3} > 0$ for $x > 0$, the sign of y matches the sign of $\ln x$, so the curve crosses the x-axis only at $x = 1$, and the enclosed region runs from $x = 1$ to $x = 3$.

Since $x^{-3} > 0$ for $x > 0$, $y = 0$ only when $\ln x = 0$, i.e. $x = 1$; for $0 < x < 1$, $\ln x < 0$ so $y < 0$, and for $x > 1$, $\ln x > 0$ so $y > 0$. Hence the curve crosses the x-axis at $x = 1$, and the region enclosed by C, the x-axis and $x = 3$ lies between $x = 1$ and $x = 3$.

$$\text{Area} = \int_1^3 x^{-3} \ln x \, dx$$

Using integration by parts with $u = \ln x$, $dv = x^{-3}dx$, so $du = \frac{1}{x}dx$, $v = -\frac{1}{2x^2}$,

$$\begin{aligned}\int x^{-3} \ln x \, dx &= -\frac{\ln x}{2x^2} - \int \left(-\frac{1}{2x^2}\right)\left(\frac{1}{x}\right) dx \\ &= -\frac{\ln x}{2x^2} + \frac{1}{2}\int x^{-3} dx \\ &= -\frac{\ln x}{2x^2} + \frac{1}{2}\left(-\frac{1}{2x^2}\right) + c \\ &= -\frac{\ln x}{2x^2} - \frac{1}{4x^2} + c\end{aligned}$$

Let $F(x) = -\frac{\ln x}{2x^2} - \frac{1}{4x^2}$. At $x = 3$, $F(3) = -\frac{\ln 3}{18} - \frac{1}{36}$; at $x = 1$, $F(1) = -\frac{\ln 1}{2} - \frac{1}{4} = 0 - \frac{1}{4} = -\frac{1}{4}$.

$$\begin{aligned}\text{Area} &= F(3) - F(1) \\ &= \left(-\frac{\ln 3}{18} - \frac{1}{36}\right) - \left(-\frac{1}{4}\right) \\ &= -\frac{\ln 3}{18} - \frac{1}{36} + \frac{1}{4}\end{aligned}$$

Using a common denominator of 36: $-\frac{1}{36} + \frac{9}{36} = \frac{8}{36} = \frac{2}{9}$, so

$$\begin{aligned}\text{Area} &= \frac{2}{9} - \frac{\ln 3}{18} \\ &= \frac{4}{18} - \frac{\ln 3}{18} \\ &= \frac{4 - \ln 3}{18}\end{aligned}$$

To check your answer: $\frac{4-\ln 3}{18} \approx \frac{4-1.0986}{18} \approx 0.1612$, matching a numerical estimate of $\int_1^3 x^{-3} \ln x \, dx \approx 0.1612$ ✓

$\therefore$ the area is $\frac{4-\ln 3}{18}$ (exact), ≈ 0.161 units^2.

Question 8 7 marks

Integration · Definite integrals

Part (a) 4 marks

APPROACH

the denominator factorises as a perfect square, so the substitution $u = x + 1$ turns the integral into a sum of two standard forms.

$$\int \frac{2x-1}{x^2+2x+1}\,dx = \int \frac{2x-1}{(x+1)^2}\,dx$$

Using the substitution $u = x + 1$, so $x = u - 1$ and $dx = du$, the numerator becomes

$$\begin{aligned} &2x - 1 \\ &= 2(u-1) - 1 \\ &= 2u - 3 \end{aligned}$$

so

$$\begin{aligned} &\int \frac{2u-3}{u^2}\,du \\ &= \int \left(\frac{2}{u} - \frac{3}{u^2}\right) du \\ &= 2\ln|u| + \frac{3}{u} + c \end{aligned}$$

Substituting back $u = x + 1$:

$$\int \frac{2x-1}{x^2+2x+1}\,dx = 2\ln|x+1| + \frac{3}{x+1} + c$$

To check your answer: differentiating, $\frac{d}{dx}\left[2\ln(x+1) + \frac{3}{x+1}\right] = \frac{2}{x+1} - \frac{3}{(x+1)^2} = \frac{2x-1}{(x+1)^2}$ ✓

$\therefore \int \frac{2x-1}{x^2+2x+1}\,dx = 2\ln|x+1| + \frac{3}{x+1} + c.$

Part (b) 3 marks

APPROACH

the modulus forces us to split the interval where $2x - 1$ changes sign, at $x = \frac{1}{2}$, and reuse the antiderivative from (a) on each piece.

Since $2x - 1 = 0$ at $x = \frac{1}{2}$: for $0 \le x < \frac{1}{2}$, $2x - 1 < 0$; for $\frac{1}{2} < x \le 2$, $2x - 1 > 0$. So

$$\int_0^2 \frac{|2x-1|}{(x+1)^2}\,dx = \int_0^{1/2} -\frac{2x-1}{(x+1)^2}\,dx + \int_{1/2}^2 \frac{2x-1}{(x+1)^2}\,dx$$

Let $F(x) = 2\ln(x+1) + \frac{3}{x+1}$; the modulus signs from part (a) can be dropped since $x + 1 > 0$ throughout $[0, 2]$. Then

$$\int_0^2 \frac{|2x-1|}{(x+1)^2}\,dx = \left[F(0) - F\left(\frac{1}{2}\right)\right] + \left[F(2) - F\left(\frac{1}{2}\right)\right]$$
$$= F(0) + F(2) - 2F\left(\frac{1}{2}\right)$$

Evaluating F at each point:

$$F(0)$$
$$= 2\ln 1 + 3$$
$$= 3$$
$$F\left(\frac{1}{2}\right)$$
$$= 2\ln\frac{3}{2} + 2$$
$$F(2)$$
$$= 2\ln 3 + 1$$

So

$$\int_0^2 \frac{|2x-1|}{(x+1)^2}\,dx$$
$$= 3 + (2\ln 3 + 1) - 2\left(2\ln\frac{3}{2} + 2\right)$$
$$= 2\ln 3 - 4\ln\frac{3}{2}$$

Using $\ln\frac{3}{2} = \ln 3 - \ln 2$:

$$2\ln 3 - 4\ln\frac{3}{2}$$
$$= 2\ln 3 - 4(\ln 3 - \ln 2)$$
$$= 4\ln 2 - 2\ln 3$$
$$= 2\ln\frac{4}{3}$$

To check your answer: $2\ln\frac{4}{3} \approx 0.575$, matching a numerical estimate of $\int_0^2 \frac{|2x-1|}{(x+1)^2}\,dx$ ✓

$\therefore \int_0^2 \frac{|2x-1|}{x^2+2x+1}\,dx = 2\ln\frac{4}{3}$.

Question 9 8 marks

Sequences and Series · Arithmetic/Geometric Progression

Part (a) 3 marks

The arithmetic series has first term a and common difference d. Its 1st, 3rd and 15th terms are

$$T_1 = a, \quad T_3 = a + 2d, \quad T_{15} = a + 14d$$

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Since these are the 1st, 2nd and 3rd terms of a geometric series, the square of the middle term equals the product of the outer two:

$$\begin{aligned} T_3^2 &= T_1 T_{15} \\ \Rightarrow (a+2d)^2 &= a(a+14d) \\ \Rightarrow a^2 + 4ad + 4d^2 &= a^2 + 14ad \\ \Rightarrow 4d^2 - 10ad &= 0 \\ \Rightarrow 2d(2d - 5a) &= 0 \end{aligned}$$

Since $d \neq 0$,

$$2d - 5a = 0$$

To check your answer: with $a = 2$, $d = 5$: $T_1 = 2$, $T_3 = 12$, $T_{15} = 72$; ratios $\frac{12}{2} = 6$ and $\frac{72}{12} = 6$ agree ✓

$\therefore d = \frac{5a}{2}$.

Part (b)(i) 3 marks

APPROACH

with a two-sided sum-to-infinity expression, the half-angle identity turns $\frac{\sin\theta}{1+\cos\theta}$ directly into a single tangent.

The geometric series has first term $\sin\theta$ and common ratio $-\cos\theta$. Since $0 < \theta < \frac{\pi}{2}$, $0 < \cos\theta < 1$, so $|-\cos\theta| < 1$ and the sum to infinity exists:

$$S_\infty = \frac{\sin\theta}{1-(-\cos\theta)} = \frac{\sin\theta}{1+\cos\theta}$$

Using the half-angle identity $\tan\left(\frac{\theta}{2}\right) = \frac{\sin\theta}{1+\cos\theta}$,

$$S_\infty = \tan\left(\frac{\theta}{2}\right) \quad \text{(shown)}$$

$\therefore k = \frac{1}{2}$.

Part (b)(ii) 2 marks

$$S_7 = \frac{43\sqrt{3}}{128}$$

Explanation: at $\theta = \frac{\pi}{3}$, the first term is $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and the common ratio is $-\cos\frac{\pi}{3} = -\frac{1}{2}$. Using $S_n = \frac{a(1-r^n)}{1-r}$ with $n = 7$,

$$\begin{aligned} S_7 &= \frac{\frac{\sqrt{3}}{2}\left(1-\left(-\frac{1}{2}\right)^7\right)}{1-\left(-\frac{1}{2}\right)} \\ &= \frac{\frac{\sqrt{3}}{2} \cdot \frac{129}{128}}{\frac{3}{2}} \end{aligned}$$

$$= \frac{\sqrt{3}}{2} \times \frac{129}{128} \times \frac{2}{3}$$
$$= \frac{129\sqrt{3}}{384}$$

Since $129 = 3 \times 43$ and $384 = 3 \times 128$, this simplifies to

$$S_7 = \frac{43\sqrt{3}}{128}$$

To check your answer: $\frac{43\sqrt{3}}{128} \approx 0.582$, matching a direct numerical sum of the first 7 terms ✓

Question 10 10 marks

Graphs and Transformations · Properties of graphs

Part (a) 3 marks

APPROACH

"no stationary points" is a statement about when $\frac{dy}{dx} = 0$ has *no* solution for $x \neq 1$, so we solve for $(x-1)^2$ and ask when that equation is unsolvable rather than searching for turning points directly.

The curve has equation

$$y = ax + b + \frac{a+2b}{x-1}, \quad a > 0,\ b \neq -\frac{1}{2}a,\ x \neq 1$$

Differentiating,

$$\frac{dy}{dx} = a - \frac{a+2b}{(x-1)^2}$$

At a stationary point, $\frac{dy}{dx} = 0$:

$$a - \frac{a+2b}{(x-1)^2} = 0 \Rightarrow (x-1)^2 = \frac{a+2b}{a}$$

Since $x \neq 1$, $(x-1)^2$ takes every value in $(0, \infty)$ as x ranges over its domain, so this equation has a solution (a stationary point exists) exactly when $\frac{a+2b}{a} > 0$, and has none when $\frac{a+2b}{a} \leq 0$. For C to have no stationary points we therefore need $\frac{a+2b}{a} \leq 0$; since $a > 0$,

$$a + 2b \leq 0$$

We are given $b \neq -\frac{1}{2}a$, i.e. $a + 2b \neq 0$, so equality is excluded:

$$a + 2b < 0$$

To check your answer: with $a = 2$, $b = -2$ (so $a + 2b = -2 < 0$): $\frac{dy}{dx} = 2 - \frac{-2}{(x-1)^2} = 2 + \frac{2}{(x-1)^2} > 0$ for all $x \neq 1$, so there is indeed no stationary point ✓

$\therefore a + 2b < 0$, i.e. $b < -\frac{1}{2}a$.

Part (b) 4 marks

> **APPROACH**
>
> before sketching we pin down every feature algebraically — both asymptotes, both intercepts — then plot.

With $b = -2a$: $a + 2b = a - 4a = -3a$, so

$$y = ax + (-2a) + \frac{-3a}{x-1} = a(x-2) - \frac{3a}{x-1}$$

Since the curve is undefined at $x = 1$, $x = 1$ is a vertical asymptote. As $x \to \pm\infty$, $\frac{3a}{x-1} \to 0$, so

$$y \to a(x-2)$$

is the oblique asymptote. At $x = 0$:

$$y = a(0-2) - \frac{3a}{0-1} = -2a + 3a = a$$

so C crosses the y-axis at $(0, a)$. Setting $y = 0$ and dividing by a (since $a > 0$):

$$(x-2) - \frac{3}{x-1} = 0$$

Multiplying through by $(x-1)$:

$$(x-2)(x-1) - 3 = 0$$

$$x^2 - 3x - 1 = 0$$

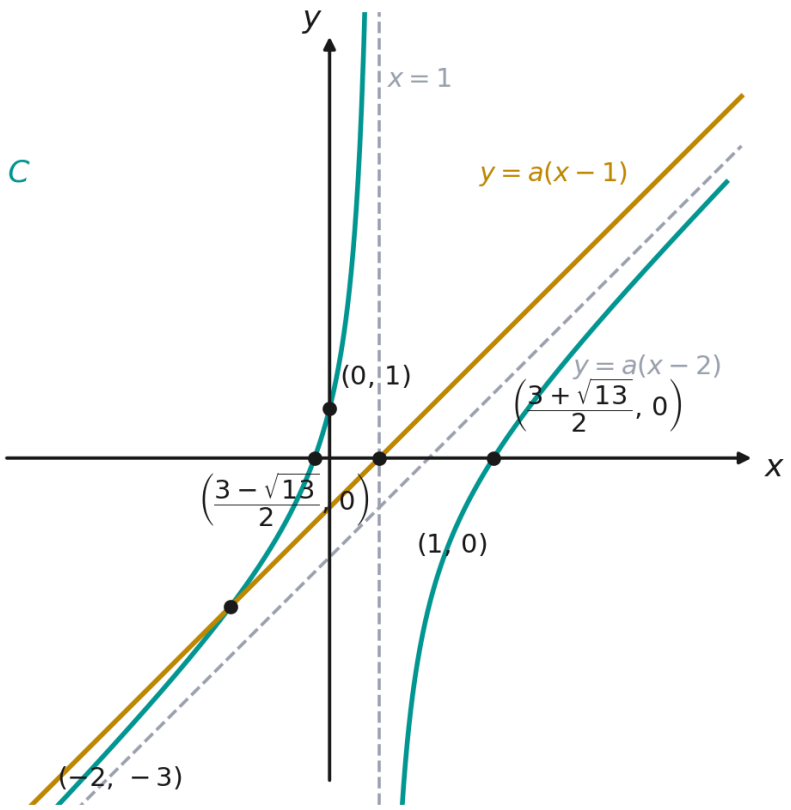
$$x = \frac{3 \pm \sqrt{13}}{2}$$

To check your answer: $x = \frac{3+\sqrt{13}}{2} \approx 3.30$; substituting into $(x-2) - \frac{3}{x-1}$ gives $\approx 1.30 - 1.30 \approx 0$ ✓

$\therefore C$ has asymptotes $x = 1$ and $y = a(x-2)$, crosses the y-axis at $(0, a)$, and crosses the x-axis at $x = \frac{3 \pm \sqrt{13}}{2}$.

The sketch below is drawn for $a = 1$ (it also shows the line from part (c), in gold).

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Part (c) 1 mark

The graph of $y = ax - a = a(x-1)$ is a straight line through $(1, 0)$ with gradient a, shown in gold on the sketch above.

Part (d) 2 marks

> **APPROACH**
>
> setting $a = 1$ turns the given inequality into exactly "curve C on or below the line", which we can read from the sketch once we know where they cross.

Setting $a = 1$ in part (b), C becomes $y = (x-2) - \frac{3}{x-1} = x - 2 - \frac{3}{x-1}$, the left side of the inequality; setting $a = 1$ in part (c), the line becomes $y = x - 1$, the right side. So solving

$$x - 2 - \frac{3}{x-1} \leq x - 1$$

is the same as finding where C (with $a = 1$) lies on or below the line $y = x - 1$.

To find where they meet, set $a(x-2) - \frac{3a}{x-1} = a(x-1)$ and divide by a:

$$(x-2) - \frac{3}{x-1} = x - 1$$

$$\Rightarrow -1 - \frac{3}{x-1} = 0$$

$$\Rightarrow \frac{3}{x-1} = -1$$

$$\Rightarrow x - 1 = -3$$

$$\Rightarrow x = -2$$

This is the only intersection (the equation reduced to a linear one, not quadratic), matching $(-2, -3)$ on the sketch.

From the sketch: for $x > 1$, the curve descends from $-\infty$ near the asymptote and stays below the line for all $x > 1$. For $x < 1$, the curve descends from $+\infty$ near the asymptote, crosses the line once at $x = -2$, and lies above the line for $-2 < x < 1$ and below it for $x < -2$.

$$\therefore\ x \leq -2 \text{ or } x > 1$$

To check your answer: at $x = -3$: LHS $= -3 - 2 - \frac{3}{-4} = -4.25$, RHS $= -4$; $-4.25 \leq -4$ ✓. At $x = 0$ (outside the solution set): LHS $= 1$, RHS $= -1$; $1 \leq -1$ is false, correctly excluded ✓. At $x = 2$: LHS $= -3$, RHS $= 1$; $-3 \leq 1$ ✓

Question 11 13 marks

Vectors · Planes

Part (a) 3 marks

Let $P = (1136, 92, p)$ and $Q = (200, 20, -15)$.

$$\overrightarrow{QP} = P - Q = (1136 - 200,\ 92 - 20,\ p - (-15)) = (936,\ 72,\ p + 15)$$

Since $PQ = 939$,

$$936^2 + 72^2 + (p + 15)^2 = 939^2$$
$$\Rightarrow 876096 + 5184 + (p + 15)^2 = 881721$$
$$\Rightarrow (p + 15)^2 = 441$$
$$\Rightarrow p + 15 = \pm 21$$
$$\Rightarrow p = 6 \text{ or } p = -36$$

Since P is below Q in level (the z-coordinate), we need $p < -15$. Only $p = -36$ satisfies this ($p = 6$ rej since $6 > -15$).

To check your answer: with $p = -36$: $\overrightarrow{QP} = (936, 72, -21)$, so $PQ = \sqrt{936^2 + 72^2 + 21^2} = \sqrt{881721} = 939$ ✓

$\therefore p = -36.$

Part (b) 4 marks

Let the three given points be $A(400, 600, -20)$, $B(500, 200, -70)$, $C(600, -340, -50)$.

$$\overrightarrow{AB} = B - A = (100, -400, -50), \quad \overrightarrow{AC} = C - A = (200, -940, -30)$$

A normal to the plane is $\overrightarrow{AB} \times \overrightarrow{AC}$:

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} (-400)(-30) - (-50)(-940) \\ (-50)(200) - (100)(-30) \\ (100)(-940) - (-400)(200) \end{pmatrix} = \begin{pmatrix} -35000 \\ -7000 \\ -14000 \end{pmatrix}$$

Dividing by -7000, a simpler normal vector is $(5, 1, 2)$. The cartesian equation of the plane is $5x + y + 2z = d$, and substituting $A(400, 600, -20)$:

$$d = 5(400) + 600 + 2(-20) = 2560$$

To check your answer: substituting $B(500, 200, -70)$: $5(500) + 200 + 2(-70) = 2560$ ✓; substituting $C(600, -340, -50)$: $5(600) - 340 + 2(-50) = 2560$ ✓

$\therefore 5x + y + 2z = 2560.$

Part (c) 4 marks

APPROACH

parametrise the line PQ, substitute into the plane equation to find the parameter at the point of intersection, and check it lies between P and Q.

The pipeline PQ has direction $\overrightarrow{PQ} = Q - P = (-936, -72, 21)$, so a vector equation of the line through P and Q is

$$\mathbf{r} = \begin{pmatrix} 1136 \\ 92 \\ -36 \end{pmatrix} + t \begin{pmatrix} -936 \\ -72 \\ 21 \end{pmatrix}, \quad t \in \mathbb{R}$$

Substituting into $5x + y + 2z = 2560$:

$$5(1136 - 936t) + (92 - 72t) + 2(-36 + 21t) = 2560$$
$$\Rightarrow 5700 - 4710t = 2560$$
$$\Rightarrow t = \frac{3140}{4710} = \frac{2}{3}$$

Since $0 < \frac{2}{3} < 1$, this point lies between P and Q, so the pipeline does pass through the rock layer. Substituting $t = \frac{2}{3}$:

$$x = 1136 - 936\left(\frac{2}{3}\right) = 512, \quad y = 92 - 72\left(\frac{2}{3}\right) = 44, \quad z = -36 + 21\left(\frac{2}{3}\right) = -22$$

To check your answer: $5(512) + 44 + 2(-22) = 2560 + 44 - 44 = 2560$ ✓, so $(512, 44, -22)$ lies on the plane

$\therefore$ the pipeline meets the rock layer at $(512, 44, -22)$.

Part (d) 2 marks

The horizontal plane has normal $\mathbf{k} = (0, 0, 1)$. The direction of the pipeline is $\mathbf{d} = (936, 72, -21)$, with $|\mathbf{d}| = 939$ (from part (a)). If θ is the angle between the pipeline and the horizontal, θ is the complement of the angle between $\mathbf{d}$ and $\mathbf{k}$, so

$$\sin\theta = \frac{|\mathbf{d}\cdot\mathbf{k}|}{|\mathbf{d}|\,|\mathbf{k}|} = \frac{|-21|}{939} = \frac{7}{313}$$
$$\Rightarrow \theta = \sin^{-1}\left(\frac{7}{313}\right) = 1.28^\circ \ (3\text{ s.f.})$$

To check your answer: $\frac{7}{313} \approx 0.0224$, and $\sin^{-1}(0.0224) \approx 1.28^\circ$; this is a small angle, consistent with the pipeline's z-displacement ($21\,m$) being much smaller than its horizontal displacement ($\approx 938.8\,m$) ✓

$\therefore \theta = 1.28^\circ$.

Question 12 13 marks

Differential Equations · Application Problems involving DEs

Part (a) 5 marks

Let k be the constant of proportionality. Since the rate of change of P is proportional to P,

$$\frac{dP}{dt} = kP$$

Separating variables and integrating,

$$\int \frac{1}{P}\,dP = \int k\,dt$$
$$\Rightarrow \ln P = kt + C$$
$$\Rightarrow P = e^{kt+C} = Ae^{kt}, \text{ where } A = e^C$$

At $t = 0$, $P = 50$:

$$50 = Ae^0 \Rightarrow A = 50$$

At $t = 10$, $P = 100$:

$$100 = 50e^{10k}$$
$$\Rightarrow e^{10k} = 2$$
$$\Rightarrow k = \frac{\ln 2}{10}$$

So the first model gives $P = 50e^{\frac{t}{10}\ln 2}$, i.e.

$$P = 50 \times 2^{t/10}$$

To check your answer: at $t = 0$, $P = 50 \times 2^0 = 50$ ✓; at $t = 10$, $P = 50 \times 2^1 = 100$ ✓

$\therefore P = 50 \times 2^{t/10}$.

Part (b) 6 marks

APPROACH

the equation $\frac{dP}{dt} = \lambda P(500 - P)$ separates, but the resulting fraction $\frac{1}{P(500-P)}$ needs partial fractions before it can be integrated.

$$\frac{dP}{dt} = \lambda P(500 - P)$$

$$\Rightarrow \int \frac{1}{P(500 - P)}\, dP = \int \lambda\, dt$$

Using partial fractions, $\frac{1}{P(500-P)} = \frac{1}{500}\left(\frac{1}{P} + \frac{1}{500-P}\right)$, so

$$\frac{1}{500}\int \left(\frac{1}{P} + \frac{1}{500 - P}\right) dP = \lambda t + C_1$$

$$\Rightarrow \ln P - \ln(500 - P) = 500\lambda t + C_2, \text{ where } C_2 = 500C_1$$

$$\Rightarrow \ln\left(\frac{P}{500 - P}\right) = 500\lambda t + C_2$$

$$\Rightarrow \frac{P}{500 - P} = Ae^{500\lambda t}, \text{ where } A = e^{C_2}$$

At $t = 0$, $P = 50$:

$$A = \frac{50}{450} = \frac{1}{9}$$

so $\frac{P}{500-P} = \frac{1}{9}e^{500\lambda t}$. At $t = 10$, $P = 100$:

$$\frac{100}{400} = \frac{1}{4} = \frac{1}{9}e^{5000\lambda}$$

$$\Rightarrow e^{5000\lambda} = \frac{9}{4}$$

$$\Rightarrow \lambda = \frac{1}{5000}\ln\left(\frac{9}{4}\right)$$

Substituting back, $e^{500\lambda t} = e^{\frac{t}{10}\ln(9/4)} = \left(\frac{9}{4}\right)^{t/10}$, so

$$\frac{P}{500 - P} = \frac{1}{9}\left(\frac{9}{4}\right)^{t/10}$$

Solving for P,

$$9P = (500 - P)\left(\frac{9}{4}\right)^{t/10}$$

$$\Rightarrow P\left[9 + \left(\frac{9}{4}\right)^{t/10}\right] = 500\left(\frac{9}{4}\right)^{t/10}$$

$$\Rightarrow P = \frac{500\left(\frac{9}{4}\right)^{t/10}}{9 + \left(\frac{9}{4}\right)^{t/10}}$$

Dividing the numerator and denominator by $\left(\frac{9}{4}\right)^{t/10}$,

$$P = \frac{500}{1 + 9\left(\frac{4}{9}\right)^{t/10}}$$

To check your answer: at $t = 0$, $P = \frac{500}{1+9} = 50$ ✓; at $t = 10$, $P = \frac{500}{1+9(4/9)} = \frac{500}{5} = 100$ ✓

$\therefore P = \frac{500}{1+9\left(\frac{4}{9}\right)^{t/10}}$.

Part (c) 2 marks

$$P \to 500 \text{ as } t \to \infty$$

Explanation: since $0 < \frac{4}{9} < 1$, $\left(\frac{4}{9}\right)^{t/10} \to 0$ as $t \to \infty$, so $P \to \frac{500}{1+0} = 500$. The refined model therefore predicts the population levels off at a fixed carrying capacity of 500. The first model, $P = 50 \times 2^{t/10}$, instead grows without bound as $t \to \infty$, which is unrealistic since real populations are constrained by limited resources, space and food supply. The refined model is an improvement because it captures this constraint.

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