TEN-YEAR SERIES · WORKED SOLUTIONS

2021 A-Level H2 Mathematics Paper 1

11 questions · 100 marks

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Question 1 5 marks

Equations · simultaneous equations · cubic coefficients

APPROACH

four conditions on $f(x) = ax^3 + bx^2 + cx + d$ give four equations in a, b, c, d. Adding and subtracting the two point conditions eliminates two unknowns at once, which is faster than substituting condition by condition.

Since the graph passes through $(1, 5)$, $f(1) = 5$:

$$a + b + c + d = 5 \quad ...(1)$$

Since the graph passes through $(-1, -3)$, $f(-1) = -3$:

$$-a + b - c + d = -3 \quad ...(2)$$

Since there is a turning point at $x = 1$, $f'(1) = 0$. Differentiating, $f'(x) = 3ax^2 + 2bx + c$:

$$3a + 2b + c = 0 \quad ...(3)$$

Since $\int_0^1 f(x)\,dx = 6$,

$$\int_0^1 \left(ax^3 + bx^2 + cx + d\right)\,dx = \left[\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx\right]_0^1$$

$$= \frac{a}{4} + \frac{b}{3} + \frac{c}{2} + d$$

so

$$\frac{a}{4} + \frac{b}{3} + \frac{c}{2} + d = 6 \quad ...(4)$$

Adding (1) and (2),

$$2b + 2d = 2$$

$$\Rightarrow b + d = 1$$

Subtracting (2) from (1),

$$2a + 2c = 8$$

$$\Rightarrow a + c = 4$$

So $d = 1 - b$ and $c = 4 - a$. Substituting into (3),

$$3a + 2b + (4 - a) = 0$$

$$\Rightarrow 2a + 2b + 4 = 0$$

$$\Rightarrow a + b = -2$$

$$\Rightarrow b = -2 - a$$

Then $d = 1 - b = 1 - (-2 - a)$:

$$d = 3 + a$$

Substituting $b = -2 - a$, $c = 4 - a$, $d = 3 + a$ into (4),

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$$\frac{a}{4}+\frac{-2-a}{3}+\frac{4-a}{2}+(3+a)=6$$

Multiplying through by 12 (the LCM of $4,3,2$),

$$3a+4(-2-a)+6(4-a)+12(3+a)=72$$
$$\Rightarrow 3a-8-4a+24-6a+36+12a=72$$
$$\Rightarrow 5a+52=72$$
$$\Rightarrow 5a=20$$
$$\Rightarrow a=4$$

Substituting back,

$$b=-2-4=-6,\quad c=4-4=0,\quad d=3+4=7$$

To check your answer: with $f(x)=4x^3-6x^2+7$: $f(1)=4-6+7=5$ ✓, $f(-1)=-4-6+7=-3$ ✓, $f'(x)=12x^2-12x$, so $f'(1)=12-12=0$ ✓, and $\int_0^1\left(4x^3-6x^2+7\right)\,dx=\left[x^4-2x^3+7x\right]_0^1=1-2+7=6$ ✓.

$\therefore a=4,\ b=-6,\ c=0,\ d=7.$

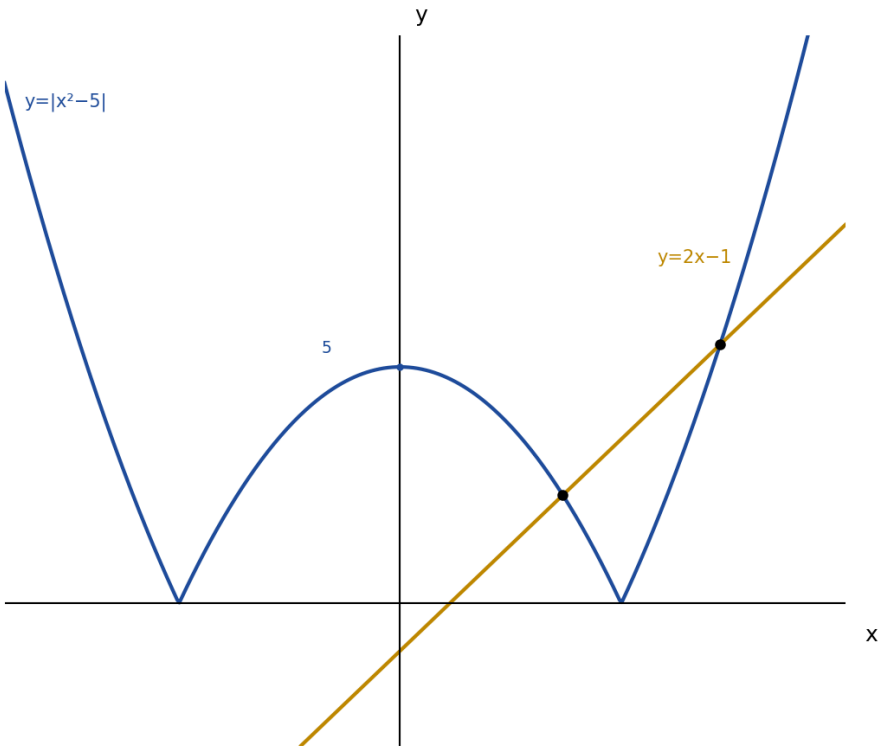
Question 2 6 marks

Graphs · modulus equations

Part (a) 2 marks

The curve $y=\left|x^2-5\right|$ meets the x-axis at $x=\pm\sqrt{5}$ and has a local maximum of 5 at $x=0$; the line $y=2x-1$ has x-intercept $\frac{1}{2}$ and y-intercept -1.

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Part (b) 4 marks

> **APPROACH**
>
> Since $\left|x^2-5\right| \geq 0$ always, a solution can only occur where $2x-1 \geq 0$. We remove the modulus by splitting into the two cases $x^2 \geq 5$ and $x^2 < 5$, solve the resulting quadratic in each case, then check every root against its case condition.

$$\left|x^2-5\right| = 2x-1$$

Since the left side is non-negative, we require $2x-1 \geq 0$, i.e. $x \geq \frac{1}{2}$, for a solution to exist.

Case 1: $x^2 \geq 5$. Here $\left|x^2-5\right| = x^2-5$, so

$$\begin{aligned} x^2-5 &= 2x-1 \\ \Rightarrow x^2-2x-4 &= 0 \\ \Rightarrow x = \frac{2 \pm \sqrt{4+16}}{2} &= 1 \pm \sqrt{5} \end{aligned}$$

Testing against $x^2 \geq 5$: $x = 1-\sqrt{5} \approx -1.24$ fails, since $\left(1-\sqrt{5}\right)^2 \approx 1.53 < 5$ (rej); $x = 1+\sqrt{5} \approx 3.24$ satisfies it.

Case 2: $x^2 < 5$. Here $\left|x^2-5\right| = 5-x^2$, so

$$\begin{aligned} 5-x^2 &= 2x-1 \\ \Rightarrow x^2+2x-6 &= 0 \\ \Rightarrow x = \frac{-2 \pm \sqrt{4+24}}{2} &= -1 \pm \sqrt{7} \end{aligned}$$

Testing against $x^2 < 5$: $x = -1-\sqrt{7} \approx -3.65$ fails, since $\left(-1-\sqrt{7}\right)^2 \approx 13.3 \geq 5$ (rej); $x = \sqrt{7}-1 \approx 1.65$ satisfies it.

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Both surviving roots also satisfy $x \geq \frac{1}{2}$.

To check your answer: at $x = 1 + \sqrt{5}$, $|x^2 - 5| = 1 + 2\sqrt{5} \approx 5.47$ and $2x - 1 \approx 5.47$ ✓; at $x = \sqrt{7} - 1$, $|x^2 - 5| = 2\sqrt{7} - 3 \approx 2.29$ and $2x - 1 \approx 2.29$ ✓.

$\therefore x = 1 + \sqrt{5}$ or $x = \sqrt{7} - 1$.

Question 3 6 marks

Differentiation · implicit differentiation

Part (a) 2 marks

$$x^{1/2} + y^{1/2} = 3$$

We differentiate implicitly with respect to x.

$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}\frac{dy}{dx} = 0$$
$$\Rightarrow \frac{1}{2}y^{-1/2}\frac{dy}{dx} = -\frac{1}{2}x^{-1/2}$$
$$\Rightarrow \frac{dy}{dx} = -\frac{x^{-1/2}}{y^{-1/2}}$$
$$= -\left(\frac{y}{x}\right)^{1/2} \quad \text{(shown)}$$

Part (b) 4 marks

Substituting $x = 1$ into the curve's equation,

$$1^{1/2} + y^{1/2} = 3$$
$$\Rightarrow y^{1/2} = 2$$
$$\Rightarrow y = 4$$

so the point is $(1, 4)$. Using the result of (a), the gradient of the curve there is

$$\frac{dy}{dx} = -\left(\frac{4}{1}\right)^{1/2} = -2$$

The normal is perpendicular to the tangent, so its gradient is the negative reciprocal, $\frac{1}{2}$. Using the point $(1, 4)$,

$$y - 4 = \frac{1}{2}(x - 1)$$
$$\Rightarrow y = \frac{1}{2}x + \frac{7}{2}$$

To check your answer: the normal's gradient $\frac{1}{2}$ times the tangent's gradient -2 gives -1, confirming they are perpendicular ✓.

$\therefore$ the normal at $x = 1$ has equation $y = \frac{1}{2}x + \frac{7}{2}$.

Question 4 7 marks

Complex numbers · polar form · powers

Part (a) 3 marks

APPROACH

No calculator is allowed, so we write each trigonometric factor as $e^{i\theta}$ using $\cos\theta + i\sin\theta = e^{i\theta}$ and combine with index laws, rather than expanding the products directly.

$$z = \frac{\left(\cos\frac{\pi}{16} + i\sin\frac{\pi}{16}\right)^2}{\cos\frac{\pi}{8} - i\sin\frac{\pi}{8}}$$

The numerator is $\left(e^{i\pi/16}\right)^2 = e^{i\pi/8}$, and the denominator is $\cos\frac{\pi}{8} - i\sin\frac{\pi}{8} = e^{-i\pi/8}$, so

$$\begin{aligned} z &= \frac{e^{i\pi/8}}{e^{-i\pi/8}} \\ &= e^{i\pi/8-(-i\pi/8)} \\ &= e^{i\pi/4} \end{aligned}$$

$\therefore |z| = 1$ and $\arg z = \frac{\pi}{4}$.

Since $z = e^{i\pi/4}$,

$$\begin{aligned} z^2 &= e^{i\pi/2} \\ &= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \\ &= i \end{aligned}$$

$\therefore z^2 = i.$

Part (b)(i) 2 marks

$$\begin{aligned} &(\cos\theta + i\sin\theta)(1 + \cos\theta - i\sin\theta) \\ &= \cos\theta + \cos^2\theta - i\cos\theta\sin\theta + i\sin\theta + i\sin\theta\cos\theta - i^2\sin^2\theta \end{aligned}$$

The terms $-i\cos\theta\sin\theta$ and $+i\sin\theta\cos\theta$ cancel, and $\cos^2\theta + \sin^2\theta = 1$ with $i^2 = -1$, leaving

$$(\cos\theta + i\sin\theta)(1 + \cos\theta - i\sin\theta) = 1 + \cos\theta + i\sin\theta \quad \text{(shown)}$$

Part (b)(ii) 2 marks

APPROACH

The identity in (b)(i), read with $\theta = \arg z$, links $z\left(1 + z^*\right)$ to $1 + z$. Raising this to the power 4 turns $(1+z)^4$ into a multiple of $\left(1 + z^*\right)^4$, so the required sum factors into $\left(1 + z^*\right)^4\left(z^4 + 1\right)$ — which is zero, since $z^4 = -1$.

Let $\theta = \arg z = \frac{\pi}{4}$, so $z = \cos\theta + i\sin\theta$ and, since $|z| = 1$, $z^* = \cos\theta - i\sin\theta$. The identity of (b)(i) then reads

$$z\left(1+z^*\right)=1+z$$

Raising both sides to the power 4,

$$(1+z)^4=z^4\left(1+z^*\right)^4$$

$$\Rightarrow (1+z)^4+\left(1+z^*\right)^4=\left(1+z^*\right)^4\left(z^4+1\right)$$

Since $z=e^{i\pi/4}$,

$$z^4=e^{i\pi}=-1$$

so $z^4+1=0$, and

$$(1+z)^4+\left(1+z^*\right)^4=\left(1+z^*\right)^4\times 0=0$$

To check your answer: direct computation with $z=e^{i\pi/4}\approx 0.707+0.707i$ gives $(1+z)^4+\left(1+z^*\right)^4\approx 0$ ✓.

$\therefore (1+z)^4+\left(1+z^*\right)^4=0.$

Question 5 7 marks

Sequences and series · method of differences

Part (a) 3 marks

$$\frac{x}{(x+2)(x+3)(x+4)}=\frac{A}{x+2}+\frac{B}{x+3}+\frac{C}{x+4}$$

Multiplying both sides by $(x+2)(x+3)(x+4)$,

$$x=A(x+3)(x+4)+B(x+2)(x+4)+C(x+2)(x+3)$$

Substituting $x=-2,\ -3,\ -4$ in turn to isolate each constant:

$x=-2$: $-2=A(1)(2)\Rightarrow A=-1$

$x=-3$: $-3=B(-1)(1)\Rightarrow B=3$

$x=-4$: $-4=C(-2)(-1)\Rightarrow C=-2$

To check your answer: at $x=1$, $\frac{1}{(3)(4)(5)}=\frac{1}{60}$, and $\frac{-1}{3}+\frac{3}{4}-\frac{2}{5}=\frac{-20+45-24}{60}=\frac{1}{60}$ ✓.

$\therefore \frac{x}{(x+2)(x+3)(x+4)}=\frac{3}{x+3}-\frac{1}{x+2}-\frac{2}{x+4}.$

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Part (b) 3 marks

APPROACH

Grouping the three partial-fraction terms as they stand does not telescope, since $\frac{-1}{r+2}$ and $\frac{-2}{r+4}$ sit two steps apart in r. Splitting $\frac{3}{r+3}$ into $\frac{1}{r+3}+\frac{2}{r+3}$ pairs each term with a neighbour exactly one step away, giving two ordinary telescoping sums.

$$\begin{aligned}\sum_{r=1}^{n}\frac{r}{(r+2)(r+3)(r+4)} &= \sum_{r=1}^{n}\left(\frac{-1}{r+2}+\frac{3}{r+3}-\frac{2}{r+4}\right)\\ &= \sum_{r=1}^{n}\left(\frac{1}{r+3}-\frac{1}{r+2}\right)+2\sum_{r=1}^{n}\left(\frac{1}{r+3}-\frac{1}{r+4}\right)\end{aligned}$$

Writing out the terms of the first sum,

$$\begin{aligned}\sum_{r=1}^{n}\left(\frac{1}{r+3}-\frac{1}{r+2}\right) &= \left(\frac{1}{4}-\frac{1}{3}\right)+\left(\frac{1}{5}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n+3}-\frac{1}{n+2}\right)\\ &= \frac{1}{n+3}-\frac{1}{3}\end{aligned}$$

and similarly, with a gap of one extra step, the second sum telescopes to

$$\sum_{r=1}^{n}\left(\frac{1}{r+3}-\frac{1}{r+4}\right)=\frac{1}{4}-\frac{1}{n+4}$$

Combining,

$$\begin{aligned}\sum_{r=1}^{n}\frac{r}{(r+2)(r+3)(r+4)} &= \left(\frac{1}{n+3}-\frac{1}{3}\right)+2\left(\frac{1}{4}-\frac{1}{n+4}\right)\\ &= \frac{1}{n+3}-\frac{2}{n+4}+\frac{1}{6}\\ &= \frac{1}{6}-\frac{n+2}{(n+3)(n+4)}\\ &= \frac{(n+3)(n+4)-6(n+2)}{6(n+3)(n+4)}\\ &= \frac{n(n+1)}{6(n+3)(n+4)}\end{aligned}$$

To check your answer: at $n=3$, direct summation gives $\frac{1}{60}+\frac{1}{60}+\frac{1}{70}\approx 0.0476$, matching $\frac{3(4)}{6(6)(7)}\approx 0.0476$ ✓.

$\therefore \sum_{r=1}^{n}\frac{r}{(r+2)(r+3)(r+4)}=\frac{n(n+1)}{6(n+3)(n+4)}$.

Part (c) 1 mark

$$\sum_{r=1}^{\infty}\frac{r}{(r+2)(r+3)(r+4)}=\frac{1}{6}$$

Explanation: dividing the numerator and denominator of the result in (b) by n^2 gives $\frac{1+1/n}{6(1+3/n)(1+4/n)}$, which tends to $\frac{1}{6}$ as $n\to\infty$.

Question 6 8 marks

Graphs and transformations · asymptotes · translation

Part (a) 4 marks

$$y = \frac{1}{\sqrt{4ax - x^2}}$$

For y to be defined, we need $4ax - x^2 > 0$, i.e. $x(4a - x) > 0$, i.e. $0 < x < 4a$.

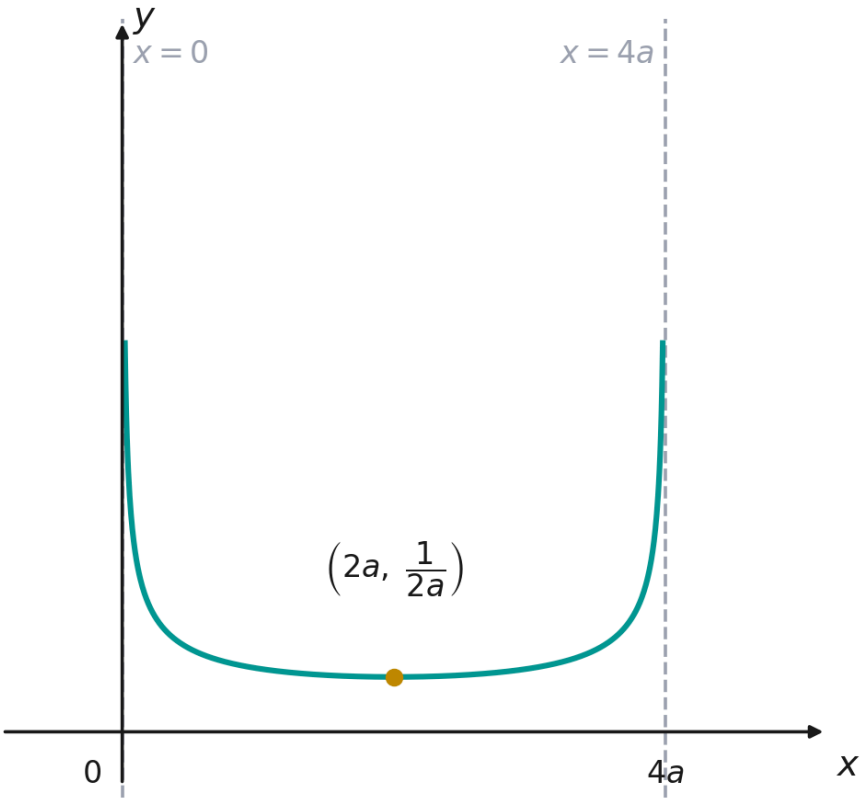
Completing the square:

$$\begin{aligned} & 4ax - x^2 \\ &= -\left(x^2 - 4ax\right) \\ &= -\left[(x - 2a)^2 - 4a^2\right] \\ &= 4a^2 - (x - 2a)^2 \end{aligned}$$

As $x \to 0^+$ or $x \to (4a)^-$, $4ax - x^2 \to 0^+$, so $y \to +\infty$: there are vertical asymptotes at $x = 0$ and $x = 4a$.

Since $4ax - x^2 = 4a^2 - (x - 2a)^2$ is maximised (equal to $4a^2$) at $x = 2a$, the denominator $\sqrt{4ax - x^2}$ is maximised there too, so y is smallest at $x = 2a$:

$$y = \frac{1}{\sqrt{4a^2}} = \frac{1}{2a}$$



To check your answer: taking $a = 1$, at $x = 2$ (the midpoint of $0 < x < 4$), $y = \frac{1}{\sqrt{8-4}} = \frac{1}{2}$, matching $\frac{1}{2a} = \frac{1}{2}$; at $x = 0.1$ or $x = 3.9$, y is much larger, consistent with the asymptotes at $x = 0$ and $x = 4a$. ✓

∴ vertical asymptotes $x = 0$ and $x = 4a$.

Part (b) 1 mark

From part (a), y is smallest at $x = 2a$.

∴ $y_{\min} = \frac{1}{2a}$.

Part (c) 3 marks

APPROACH

From part (a), C's equation is already written in completed-square form, $y = \frac{1}{\sqrt{4a^2-(x-2a)^2}}$, so we can read the transformation off by comparing this directly with the target equation, without differentiating or solving anything.

Let $g(x) = \frac{1}{\sqrt{4a^2-x^2}}$ be the target curve. From part (a), C's equation is $y = g(x-2a)$.

Since $g(x-2a)$ is the graph of g shifted $2a$ units in the positive x-direction, C is g translated $2a$ units right. Reversing this, g is obtained from C by a translation $2a$ units in the negative x-direction.

To check your answer: C is defined on $0 < x < 4a$ with minimum point $\left(2a, \frac{1}{2a}\right)$; translating $2a$ units in the negative x-direction gives domain $-2a < x < 2a$ with minimum point $\left(0, \frac{1}{2a}\right)$. Directly, $y = \frac{1}{\sqrt{4a^2-x^2}}$ is indeed defined on $-2a < x < 2a$, with minimum value $\frac{1}{\sqrt{4a^2}} = \frac{1}{2a}$ at $x = 0$. ✓

∴ a translation of $2a$ units in the negative x-direction.

Question 7 9 marks

Maclaurin series · repeated differentiation

Part (a) 4 marks

$$y = e^{\sin^{-1} x}$$

Differentiating using the chain rule, with $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$:

$$\frac{dy}{dx} = e^{\sin^{-1} x} \times \frac{1}{\sqrt{1-x^2}}$$
$$= \frac{y}{\sqrt{1-x^2}}$$

APPROACH

this relates $\frac{dy}{dx}$ to y, but the target equation has $\frac{d^2y}{dx^2}$, so we need a second derivative. To avoid differentiating $\frac{1}{\sqrt{1-x^2}}$ directly, we first clear the square root by squaring, then differentiate implicitly.

Rearranging and squaring both sides:

$$\sqrt{1-x^2}\,\frac{dy}{dx} = y$$
$$\Rightarrow \left(1-x^2\right)\left(\frac{dy}{dx}\right)^2 = y^2$$

Differentiating both sides implicitly with respect to x (product rule on the left, chain rule on the right):

$$-2x\left(\frac{dy}{dx}\right)^2 + \left(1-x^2\right) \times 2\frac{dy}{dx}\frac{d^2y}{dx^2} = 2y\frac{dy}{dx}$$

Dividing throughout by $2\frac{dy}{dx}$, which is non-zero since $\frac{dy}{dx} = \frac{y}{\sqrt{1-x^2}}$ and $y = e^{\sin^{-1}x} > 0$:

$$-x\frac{dy}{dx} + \left(1-x^2\right)\frac{d^2y}{dx^2} = y$$
$$\left(1-x^2\right)\frac{d^2y}{dx^2} = x\frac{dy}{dx} + y \quad \text{(shown)}$$

To check your answer: at $x = 0$, $y = e^{\sin^{-1}0} = 1$ and $\frac{dy}{dx} = \frac{y}{\sqrt{1-x^2}} = 1$; substituting into the shown identity gives $\frac{d^2y}{dx^2} = 0(1) + 1 = 1$ at $x = 0$, a finite, sensible value. ✓

Part (b) 5 marks

At $x = 0$: $y(0) = e^{\sin^{-1}0} = 1$.

From $\frac{dy}{dx} = \frac{y}{\sqrt{1-x^2}}$, at $x = 0$: $y'(0) = 1$.

From the result of part (a), $\left(1-x^2\right)y'' = xy' + y$; at $x = 0$:

$$y''(0) = 0 \times y'(0) + y(0) = 1$$

Differentiating $\left(1-x^2\right)y'' = xy' + y$ with respect to x (product rule on both sides):

$$-2xy'' + \left(1-x^2\right)y''' = y' + xy'' + y'$$
$$= 2y' + xy''$$
$$\Rightarrow \left(1-x^2\right)y''' = 2y' + 3xy''$$

At $x = 0$:

$$y'''(0) = 2y'(0) + 0 = 2$$

Using the Maclaurin series $y = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \cdots$:

$$y = 1 + x + \frac{1}{2}x^2 + \frac{2}{6}x^3 + \cdots$$
$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$$

To check your answer: at $x = 0.1$, $e^{\sin^{-1}(0.1)} \approx 1.10536$, and $1 + 0.1 + \frac{1}{2}(0.01) + \frac{1}{3}(0.001) \approx 1.10533$, a close match. ✓

$\therefore e^{\sin^{-1} x} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$

Question 8 14 marks

Vectors · skew lines · shortest distance

Part (a) 4 marks

APPROACH

a plane containing l_1 and the point P is fixed once we have two directions lying in it — the line's direction $\mathbf{d}_1$ and $\overrightarrow{AP}$, where A is a point on l_1 — so we find the normal as their cross product.

l_1 has direction $\mathbf{d}_1 = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ and passes through $A = (3, 2, 0)$. The plane also contains $P = (1, -3, -1)$, so

$$\overrightarrow{AP} = P - A = \begin{pmatrix} -2 \\ -5 \\ -1 \end{pmatrix}$$

A normal to the plane is $\mathbf{d}_1 \times \overrightarrow{AP}$:

$$\mathbf{d}_1 \times \overrightarrow{AP} = \begin{pmatrix} (-3)(-1) - (1)(-5) \\ (1)(-2) - (2)(-1) \\ (2)(-5) - (-3)(-2) \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ -16 \end{pmatrix}$$

Dividing by 8, a simpler normal vector is $\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$. The cartesian equation of the plane is $x - 2z = d$, and substituting $A = (3, 2, 0)$:

$$d = 3 - 2(0) = 3$$

To check your answer: substituting $P = (1, -3, -1)$: $1 - 2(-1) = 3$, matching the plane's constant. ✓

$\therefore$ the plane has cartesian equation $x - 2z = 3$.

Part (b) 2 marks

l_1 has direction $\mathbf{d}_1 = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ and l_2 has direction $\mathbf{d}_2 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$.

$$\mathbf{d}_1 \cdot \mathbf{d}_2 = (2)(1) + (-3)(2) + (1)(4) = 2 - 6 + 4 = 0$$

Since the dot product of their direction vectors is 0, $l_1 \perp l_2$ (shown).

Part (c)(i) 6 marks

> **APPROACH**
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> the common perpendicular between two skew lines meets each at a specific parameter value. We find $\mathbf{r}_1 - \mathbf{r}_2$ in terms of λ and μ, then impose perpendicularity to both direction vectors as two simultaneous equations.

$$\mathbf{r}_1 - \mathbf{r}_2 = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} - \begin{pmatrix} -4 \\ 1 \\ -3 \end{pmatrix} - \mu \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 7 + 2\lambda - \mu \\ 1 - 3\lambda - 2\mu \\ 3 + \lambda - 4\mu \end{pmatrix}$$

For $\mathbf{r}_1 - \mathbf{r}_2$ to be perpendicular to $\mathbf{d}_1$:

$$\begin{aligned} 2(7 + 2\lambda - \mu) - 3(1 - 3\lambda - 2\mu) + 1(3 + \lambda - 4\mu) &= 0 \\ \Rightarrow 14 + 14\lambda + 0\mu &= 0 \\ \Rightarrow \lambda &= -1 \end{aligned}$$

For $\mathbf{r}_1 - \mathbf{r}_2$ to be perpendicular to $\mathbf{d}_2$:

$$\begin{aligned} 1(7 + 2\lambda - \mu) + 2(1 - 3\lambda - 2\mu) + 4(3 + \lambda - 4\mu) &= 0 \\ \Rightarrow 21 + 0\lambda - 21\mu &= 0 \\ \Rightarrow \mu &= 1 \end{aligned}$$

To check your answer: with $\lambda = -1,\ \mu = 1$: $\mathbf{r}_1 - \mathbf{r}_2 = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}$, and $\begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix} \cdot \mathbf{d}_1 = 8 - 6 - 2 = 0$, $\begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix} \cdot \mathbf{d}_2 = 4 + 4 - 8 = 0$. ✓

Substituting back into the line equations, the common perpendicular meets l_1 at

$$\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} + (-1) \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix}$$

and meets l_2 at

$$\begin{pmatrix} -4 \\ 1 \\ -3 \end{pmatrix} + (1) \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix}$$

$\therefore \lambda = -1$, $\mu = 1$; the common perpendicular meets l_1 at $\begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix}$ and l_2 at $\begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix}$.

Part (c)(ii) 2 marks

The length of the common perpendicular is the distance between the two meeting points found in (i).

$$\left|\begin{pmatrix}1\\5\\-1\end{pmatrix}-\begin{pmatrix}-3\\3\\1\end{pmatrix}\right|$$
$$=\left|\begin{pmatrix}4\\2\\-2\end{pmatrix}\right|$$
$$=\sqrt{4^2+2^2+(-2)^2}$$
$$=\sqrt{24}$$
$$=2\sqrt{6}$$

To check your answer: $2\sqrt{6}\approx 4.90$; since $\begin{pmatrix}4\\2\\-2\end{pmatrix}$ is genuinely perpendicular to both line directions (confirmed in (i)), this is indeed the shortest distance between the two skew lines. ✓

$\therefore$ the length of the common perpendicular is $2\sqrt{6}$ units.

Question 9 13 marks

Integration · stationary points · volume of revolution

Part (a) 5 marks

$$f(x)=e^x\cos x$$

Using the product rule:

$$f'(x)=e^x\cos x-e^x\sin x=e^x(\cos x-\sin x)$$

At a stationary point, $f'(x)=0$. Since $e^x\neq 0$:

$$\cos x-\sin x=0$$
$$\Rightarrow \tan x=1$$
$$\Rightarrow x=\frac{\pi}{4}$$

(within the given domain $0\leq x\leq \frac{\pi}{2}$).

Substituting into $f(x)$:

$$f\left(\frac{\pi}{4}\right)=e^{\pi/4}\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}e^{\pi/4}$$

To determine the nature, we differentiate again:

$$f''(x)=\frac{d}{dx}\left[e^x(\cos x-\sin x)\right]$$
$$=e^x(\cos x-\sin x)+e^x(-\sin x-\cos x)$$
$$=-2e^x\sin x$$

At $x=\frac{\pi}{4}$:

$$f''\left(\frac{\pi}{4}\right)=-2e^{\pi/4}\sin\frac{\pi}{4}=-\sqrt{2}\,e^{\pi/4}<0$$

Since $f''\left(\frac{\pi}{4}\right) < 0$, this stationary point is a maximum.

To check your answer: $f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}e^{\pi/4} \approx 1.551$; nearby, $f(0.7) \approx 1.548$ and $f(0.85) \approx 1.545$, both slightly less, confirming a local maximum. ✓

∴ stationary point $\left(\frac{\pi}{4},\ \frac{\sqrt{2}}{2}e^{\pi/4}\right)$, a maximum.

Part (b) 4 marks

APPROACH

integrating by parts once returns an integral of the same "$e^{2x} \times$ trig" type, not a simpler one, so we name the second integral J, apply parts to it as well, and solve the resulting equation for I once J is expressed back in terms of I.

Let $I = \int e^{2x} \cos 2x\, dx$. Using integration by parts with $u = e^{2x}$, $dv = \cos 2x\, dx$, so $du = 2e^{2x}\, dx$, $v = \frac{1}{2}\sin 2x$:

$$I = \frac{1}{2}e^{2x}\sin 2x - \int e^{2x}\sin 2x\, dx$$

Let $J = \int e^{2x}\sin 2x\, dx$. Using integration by parts again with $u = e^{2x}$, $dv = \sin 2x\, dx$, so $du = 2e^{2x}\, dx$, $v = -\frac{1}{2}\cos 2x$:

$$J = -\frac{1}{2}e^{2x}\cos 2x + \int e^{2x}\cos 2x\, dx = -\frac{1}{2}e^{2x}\cos 2x + I$$

Substituting back into the expression for I:

$$\begin{aligned} I &= \frac{1}{2}e^{2x}\sin 2x - J \\ &= \frac{1}{2}e^{2x}\sin 2x - \left(-\frac{1}{2}e^{2x}\cos 2x + I\right) \\ &= \frac{1}{2}e^{2x}\sin 2x + \frac{1}{2}e^{2x}\cos 2x - I \\ 2I &= \frac{1}{2}e^{2x}\left(\sin 2x + \cos 2x\right) \\ \Rightarrow I &= \frac{1}{4}e^{2x}\left(\sin 2x + \cos 2x\right) + c \end{aligned}$$

To check your answer: differentiating $\frac{1}{4}e^{2x}\left(\sin 2x + \cos 2x\right)$ gives $\frac{1}{4}\left[2e^{2x}\left(\sin 2x + \cos 2x\right) + e^{2x}\left(2\cos 2x - 2\sin 2x\right)\right] = e^{2x}\cos 2x$. ✓

∴ $\int e^{2x}\cos 2x\, dx = \frac{1}{4}e^{2x}\left(\sin 2x + \cos 2x\right) + c$ (shown).

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Part (c) 4 marks

APPROACH

rotating $y = f(x)$ about the x-axis gives $V = \pi \int y^2\,dx$ with $y^2 = e^{2x}\cos^2 x$. Since part (b) integrates $e^{2x}\cos 2x$, not $e^{2x}\cos^2 x$, we use $\cos^2 x = \frac{1+\cos 2x}{2}$ to split the integral into a piece we can integrate directly and a piece matching part (b).

$$V = \pi \int_0^{\pi/2} [f(x)]^2\,dx = \pi \int_0^{\pi/2} e^{2x}\cos^2 x\,dx$$

Using $\cos^2 x = \frac{1+\cos 2x}{2}$:

$$V = \frac{\pi}{2}\int_0^{\pi/2} e^{2x}\,dx + \frac{\pi}{2}\int_0^{\pi/2} e^{2x}\cos 2x\,dx$$

For the first integral:

$$\int_0^{\pi/2} e^{2x}\,dx = \left[\frac{1}{2}e^{2x}\right]_0^{\pi/2} = \frac{1}{2}\left(e^{\pi} - 1\right)$$

For the second integral, using the result of part (b):

$$\int_0^{\pi/2} e^{2x}\cos 2x\,dx = \left[\frac{1}{4}e^{2x}\left(\sin 2x + \cos 2x\right)\right]_0^{\pi/2}$$

At $x = \frac{\pi}{2}$: $\sin\pi = 0$, $\cos\pi = -1$, giving $\frac{1}{4}e^{\pi}(0-1) = -\frac{1}{4}e^{\pi}$.

At $x = 0$: $\sin 0 = 0$, $\cos 0 = 1$, giving $\frac{1}{4}(1)(0+1) = \frac{1}{4}$.

$$\int_0^{\pi/2} e^{2x}\cos 2x\,dx = -\frac{1}{4}e^{\pi} - \frac{1}{4} = -\frac{1}{4}\left(e^{\pi} + 1\right)$$

Substituting back:

$$\begin{aligned} V &= \frac{\pi}{2} \times \frac{1}{2}\left(e^{\pi} - 1\right) + \frac{\pi}{2} \times \left[-\frac{1}{4}\left(e^{\pi} + 1\right)\right] \\ &= \frac{\pi}{4}\left(e^{\pi} - 1\right) - \frac{\pi}{8}\left(e^{\pi} + 1\right) \end{aligned}$$

Combining over a common denominator of 8:

$$\begin{aligned} V &= \frac{2\pi\left(e^{\pi} - 1\right) - \pi\left(e^{\pi} + 1\right)}{8} \\ &= \frac{\pi\left[2e^{\pi} - 2 - e^{\pi} - 1\right]}{8} \\ &= \frac{\pi\left(e^{\pi} - 3\right)}{8} \end{aligned}$$

To check your answer: $\frac{\pi(e^{\pi}-3)}{8} \approx 7.91$, matching a direct numerical estimate of $\pi\int_0^{\pi/2} e^{2x}\cos^2 x\,dx$. ✓

$\therefore V = \frac{\pi(e^{\pi}-3)}{8}$ units^3.

Question 10 12 marks

Differential equations · bacteria growth model

Part (a) 4 marks

Since the growth rate is proportional to N,

$$\frac{dN}{dt} = kN$$

Separating variables and integrating,

$$\int \frac{1}{N}\, dN = \int k\, dt$$
$$\Rightarrow \ln N = kt + c$$
$$\Rightarrow N = Ae^{kt}, \text{ where } A = e^c$$

At $t = 0$, $N = 100$:

$$100 = Ae^0$$
$$\Rightarrow A = 100$$

At $t = 2$, $N = 300$:

$$300 = 100e^{2k}$$
$$\Rightarrow e^{2k} = 3$$
$$\Rightarrow k = \frac{1}{2}\ln 3$$

So $N = 100e^{\frac{1}{2}t\ln 3} = 100 \times 3^{t/2}$.

To find when $N = 1000$,

$$1000 = 100 \times 3^{t/2}$$
$$\Rightarrow 3^{t/2} = 10$$
$$\Rightarrow \frac{t}{2}\ln 3 = \ln 10$$
$$\Rightarrow t = \frac{2\ln 10}{\ln 3}$$

To check your answer: $t = \frac{2\ln 10}{\ln 3} \approx 4.192$ min; at this t, $100 \times 3^{t/2} = 100 \times 3^{2.096} \approx 1000$ ✓

$\therefore k = \frac{1}{2}\ln 3$, and the number of bacteria reaches 1000 after $t = \frac{2\ln 10}{\ln 3} \approx 4.19$ minutes.

Part (b) 5 marks

Now the growth rate is reduced by d bacteria per minute, so

$$\frac{dN}{dt} = kN - d$$

Separating variables and integrating,

$$\int \frac{1}{kN - d}\, dN = \int dt$$

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$$\Rightarrow \frac{1}{k}\ln|kN-d| = t + c$$

At $t = 0$, $N = 100$:

$$0 = \frac{1}{k}\ln|100k-d| + c$$
$$\Rightarrow c = -\frac{1}{k}\ln|100k-d|$$

Substituting back,

$$t = \frac{1}{k}\ln|kN-d| - \frac{1}{k}\ln|100k-d|$$
$$= \frac{1}{k}\ln\left|\frac{kN-d}{100k-d}\right|$$

Now we make N the subject.

$$kt = \ln\left|\frac{kN-d}{100k-d}\right|$$
$$\Rightarrow e^{kt} = \frac{kN-d}{100k-d}$$
$$\Rightarrow kN - d = (100k-d)e^{kt}$$
$$\Rightarrow N = \left(100 - \frac{d}{k}\right)e^{kt} + \frac{d}{k}$$

To check your answer: at $t = 0$, $N = \left(100 - \frac{d}{k}\right)(1) + \frac{d}{k} = 100$ ✓; differentiating gives $\frac{dN}{dt} = k\left(100 - \frac{d}{k}\right)e^{kt} = kN - d$, matching the original equation ✓

$\therefore t = \frac{1}{k}\ln\left|\frac{kN-d}{100k-d}\right|$ and $N = \left(100 - \frac{d}{k}\right)e^{kt} + \frac{d}{k}$, where $k = \frac{1}{2}\ln 3$.

Part (c)(i) 1 mark

$$d > 50\ln 3$$

Explanation: writing $N(t) = \left(100 - \frac{d}{k}\right)e^{kt} + \frac{d}{k}$, the coefficient of the growing term e^{kt} is $100 - \frac{d}{k}$. If $d > 100k$, this coefficient is negative, so $\frac{dN}{dt} = k\left(100 - \frac{d}{k}\right)e^{kt}$ is negative for every t, and N decreases throughout. If $d \leq 100k$, the coefficient is non-negative and N does not decrease. Since $100k = 50\ln 3$, the number of bacteria decreases if and only if $d > 50\ln 3$.

$\therefore$ the number of bacteria decreases for $d > 50\ln 3$ (≈ 54.9).

Part (c)(ii) 2 marks

Since $d = 58 > 50\ln 3 \approx 54.93$, the population does decrease to zero. Setting $N = 0$ in the formula from part (b),

$$0 = \left(100 - \frac{58}{k}\right)e^{kt} + \frac{58}{k}$$
$$\Rightarrow e^{kt} = \frac{58}{58 - 100k}$$

$$\Rightarrow t = \frac{1}{k} \ln\left(\frac{58}{58 - 100k}\right)$$

Substituting $k = \frac{1}{2}\ln 3$,

$$t = \frac{2}{\ln 3} \ln\left(\frac{58}{58 - 50\ln 3}\right)$$

To check your answer: $58 - 50\ln 3 \approx 3.069$, so $\frac{58}{3.069} \approx 18.90$ and $t \approx \frac{2}{1.0986}\ln(18.90) \approx 5.350$ min; substituting this t back into $N(t)$ with $d = 58$ gives $N \approx 0$ ✓

$\therefore t = \frac{2}{\ln 3} \ln\left(\frac{58}{58 - 50\ln 3}\right) \approx 5.35$ minutes.

Question 11 13 marks

Parametric equations · area · greatest and least values

Part (a) 2 marks

Since PQ is parallel to the y-axis, P has the same x-coordinate as Q, i.e. at $\theta = \beta$:

$$x_P = a(\beta - \sin\beta)$$

Similarly S has the same x-coordinate as R, i.e. at $\theta = 2\pi - \beta$:

$$x_S = a[(2\pi - \beta) - \sin(2\pi - \beta)]$$

Since $\sin(2\pi - \beta) = -\sin\beta$,

$$= a(2\pi - \beta + \sin\beta)$$

Then

$$PS = x_S - x_P$$
$$= a(2\pi - \beta + \sin\beta) - a(\beta - \sin\beta)$$
$$= 2a(\pi - \beta + \sin\beta)$$

$\therefore PS = 2a(\pi - \beta + \sin\beta)$.

Part (b) 6 marks

APPROACH

The shaded area is $\int y\,dx$ between P and S. Converting to the parameter θ via $dx = \frac{dx}{d\theta}\,d\theta$ turns this into $\int_\beta^{2\pi-\beta} y\,\frac{dx}{d\theta}\,d\theta$, which reduces to $\int (1-\cos\theta)^2\,d\theta$ — expand this using $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$ before integrating, since $(1-\cos\theta)^2$ itself has no elementary antiderivative.

Since $\frac{dx}{d\theta} = a(1-\cos\theta)$ and $y = a(1-\cos\theta)$,

$$\text{Area} = \int_\beta^{2\pi-\beta} y\,\frac{dx}{d\theta}\,d\theta$$

$$= \int_{\beta}^{2\pi-\beta} a^2 (1-\cos\theta)^2 \, d\theta$$

Expanding, and using $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$,

$$(1-\cos\theta)^2 = 1 - 2\cos\theta + \cos^2\theta$$
$$= \frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta$$

So

$$\text{Area} = a^2 \int_{\beta}^{2\pi-\beta} \left(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta$$
$$= a^2 \left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_{\beta}^{2\pi-\beta}$$

Using $\sin(2\pi-\beta) = -\sin\beta$ and $\sin(4\pi-2\beta) = -\sin 2\beta$: at $\theta = 2\pi-\beta$ the bracket is $3\pi - \frac{3}{2}\beta + 2\sin\beta - \frac{1}{4}\sin 2\beta$, and at $\theta = \beta$ it is $\frac{3}{2}\beta - 2\sin\beta + \frac{1}{4}\sin 2\beta$. Subtracting,

$$\text{Area} = a^2 \left[\left(3\pi - \frac{3}{2}\beta + 2\sin\beta - \frac{1}{4}\sin 2\beta\right) - \left(\frac{3}{2}\beta - 2\sin\beta + \frac{1}{4}\sin 2\beta\right)\right]$$
$$= a^2 \left(3\pi - 3\beta + 4\sin\beta - \frac{1}{2}\sin 2\beta\right)$$
$$= \frac{1}{2}a^2 (6\pi - 6\beta + 8\sin\beta - \sin 2\beta)$$

To check your answer: at $\beta = 0$ (degenerate case, Q and R both at the base points of the cycloid), Area $= \frac{1}{2}a^2(6\pi) = 3\pi a^2$, matching the known area under one full cycloid arch ✓

$\therefore$ Area $= \frac{1}{2}a^2 (6\pi - 6\beta + 8\sin\beta - \sin 2\beta)$ (shown).

Part (c) 1 mark

$$\beta \approx 1.90$$

Explanation: dividing the area equation $\frac{1}{2}a^2 (6\pi - 6\beta + 8\sin\beta - \sin 2\beta) = 7.8159a^2$ through by a^2 leaves a transcendental equation in β alone,

$$3\pi - 3\beta + 4\sin\beta - \frac{1}{2}\sin 2\beta = 7.8159$$

which has no algebraic solution. By GC, $\beta = 1.90$ (3 s.f.).

$\therefore \beta = 1.90$.

Part (d) 4 marks

APPROACH

With β now known, part (a) gives a from $PS = 50$. The greatest and least heights of the arch above the expressway are then the greatest and least values of $y = a(1-\cos\theta)$ for θ between β and $2\pi - \beta$.

From part (a), $50 = 2a\,(\pi - \beta + \sin\beta)$. Substituting $\beta = 1.90$,

$$a = \frac{25}{\pi - 1.90 + \sin 1.90} \approx 11.4$$

Since $\frac{dy}{d\theta} = a\sin\theta$, y is increasing for $0 < \theta < \pi$ and decreasing for $\pi < \theta < 2\pi$. As $\beta \approx 1.90 < \pi < 2\pi - \beta$, on the arch QR the height y rises from $\theta = \beta$ to a maximum at $\theta = \pi$, then falls back to the same value at $\theta = 2\pi - \beta$ (since $\cos(2\pi - \beta) = \cos\beta$, Q and R are at equal height).

Greatest height, at the apex $\theta = \pi$:

$$y_{\max} = a\,(1 - \cos\pi) = 2a \approx 22.9\ m$$

Least height, at Q and R, $\theta = \beta$:

$$y_{\min} = a\,(1 - \cos\beta) = a\,(1 - \cos 1.90) \approx 15.1\ m$$

To check your answer: $y_{\min} \approx 15.1\ m < y_{\max} \approx 22.9\ m$, as expected for an arch rising from its supports to a central peak ✓

$\therefore$ the greatest height of the arch is $\approx 22.9\ m$ and the least height is $\approx 15.1\ m$.

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