TEN-YEAR SERIES · WORKED SOLUTIONS

2020 A-Level H2 Mathematics Paper 1

11 questions · 100 marks

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Physics. Math. Chemistry.

Question 1 4 marks

Vectors · planes · normals

Part (i) 2 marks

We find a normal to π_1 using the cross product of the two vectors lying in the plane.

$$\begin{pmatrix}1\\1\\0\end{pmatrix} \times \begin{pmatrix}1\\-5\\-2\end{pmatrix}$$
$$= \begin{pmatrix}(1)(-2)-(0)(-5)\\(0)(1)-(1)(-2)\\(1)(-5)-(1)(1)\end{pmatrix}$$
$$= \begin{pmatrix}-2\\2\\-6\end{pmatrix}$$

Dividing by -2 to simplify,

$$\begin{pmatrix}-2\\2\\-6\end{pmatrix} = -2\begin{pmatrix}1\\-1\\3\end{pmatrix}$$

To check your answer: $\begin{pmatrix}1\\-1\\3\end{pmatrix}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix} = 1-1+0=0$ and $\begin{pmatrix}1\\-1\\3\end{pmatrix}\cdot\begin{pmatrix}1\\-5\\-2\end{pmatrix} = 1+5-6=0$, confirming the vector is perpendicular to both vectors in the plane. ✓

$\therefore$ a normal to π_1 is $\begin{pmatrix}1\\-1\\3\end{pmatrix}$.

Part (ii) 2 marks

APPROACH

The acute angle between two planes equals the acute angle between their normal vectors, found with the dot-product formula $\cos\theta = \frac{|\mathbf{n}_1\cdot\mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|}$.

A normal to π_2 is $\begin{pmatrix}4\\5\\-6\end{pmatrix}$, read directly from its cartesian equation. Let θ be the acute angle between π_1 and π_2.

$$\cos\theta = \frac{\left|\begin{pmatrix}1\\-1\\3\end{pmatrix}\cdot\begin{pmatrix}4\\5\\-6\end{pmatrix}\right|}{\left|\begin{pmatrix}1\\-1\\3\end{pmatrix}\right|\left|\begin{pmatrix}4\\5\\-6\end{pmatrix}\right|}$$

We compute the dot product and the two magnitudes.

$$\begin{pmatrix}1\\-1\\3\end{pmatrix}\cdot\begin{pmatrix}4\\5\\-6\end{pmatrix}=4-5-18=-19$$

$$\left|\begin{pmatrix}1\\-1\\3\end{pmatrix}\right|=\sqrt{11},\quad\left|\begin{pmatrix}4\\5\\-6\end{pmatrix}\right|=\sqrt{77}$$

So

$$\begin{aligned}\cos\theta&=\frac{19}{\sqrt{11}\times\sqrt{77}}\\&=\frac{19}{\sqrt{847}}\\&=\frac{19\sqrt{7}}{77}\\&\approx 0.6528\\\theta&\approx 49.2^\circ\ (1\text{ d.p.})\end{aligned}$$

To check your answer: $\theta\approx 49.2^\circ$ lies between 0° and 90°, confirming it is the acute angle between the planes. ✓

$\therefore$ the acute angle between π_1 and π_2 is 49.2°.

Question 2 6 marks

Differentiation · implicit · tangents

We differentiate both sides of $\frac{x^2}{1+x^2}+\frac{y^2}{1+y^2}=x^3y^5$ implicitly with respect to x.

For the left side, we apply the quotient rule to each term.

$$\begin{aligned}&\frac{d}{dx}\left(\frac{x^2}{1+x^2}\right)\\&=\frac{2x\left(1+x^2\right)-x^2(2x)}{\left(1+x^2\right)^2}\\&=\frac{2x}{\left(1+x^2\right)^2}\end{aligned}$$

$$\frac{d}{dx}\left(\frac{y^2}{1+y^2}\right)=\frac{2y\frac{dy}{dx}\left(1+y^2\right)-y^2\left(2y\frac{dy}{dx}\right)}{\left(1+y^2\right)^2}=\frac{2y\frac{dy}{dx}}{\left(1+y^2\right)^2}$$

For the right side, we apply the product rule.

$$\frac{d}{dx}\left(x^3y^5\right)=3x^2y^5+5x^3y^4\frac{dy}{dx}$$

So

$$\frac{2x}{\left(1+x^2\right)^2}+\frac{2y}{\left(1+y^2\right)^2}\frac{dy}{dx}=3x^2y^5+5x^3y^4\frac{dy}{dx}$$

To check your answer: substituting $x = 1$, $y = 1$ into the original equation, $\frac{1}{2} + \frac{1}{2} = 1$ and $x^3y^5 = 1$, so $(1, 1)$ does lie on the curve. ✓

Substituting $x = 1$, $y = 1$,

$$\frac{2}{4} + \frac{2}{4}\frac{dy}{dx} = 3 + 5\frac{dy}{dx}$$
$$\Rightarrow \frac{1}{2} + \frac{1}{2}\frac{dy}{dx} = 3 + 5\frac{dy}{dx}$$
$$\Rightarrow \frac{1}{2} - 3 = 5\frac{dy}{dx} - \frac{1}{2}\frac{dy}{dx}$$
$$\Rightarrow -\frac{5}{2} = \frac{9}{2}\frac{dy}{dx}$$
$$\frac{dy}{dx} = -\frac{5}{2} \times \frac{2}{9} = -\frac{5}{9}$$

The tangent at $(1, 1)$ with gradient $-\frac{5}{9}$ is

$$y - 1 = -\frac{5}{9}(x - 1)$$
$$\Rightarrow 9(y - 1) = -5(x - 1)$$
$$\Rightarrow 9y - 9 = -5x + 5$$
$$\Rightarrow 5x + 9y = 14$$

$\therefore$ the tangent has equation $5x + 9y = 14$.

Question 3 7 marks

Maclaurin series · repeated differentiation

Part (i) 3 marks

We differentiate $f(x) = \ln(1 + \sin 3x)$ using the chain rule.

$$f'(x) = \frac{3\cos 3x}{1 + \sin 3x}$$

Differentiating again using the quotient rule, with numerator $u = 3\cos 3x$ (so $u' = -9\sin 3x$) and denominator $v = 1 + \sin 3x$ (so $v' = 3\cos 3x$),

$$f''(x) = \frac{u'v - uv'}{v^2}$$
$$= \frac{(-9\sin 3x)(1 + \sin 3x) - (3\cos 3x)(3\cos 3x)}{(1 + \sin 3x)^2}$$
$$= \frac{-9\sin 3x - 9\sin^2(3x) - 9\cos^2(3x)}{(1 + \sin 3x)^2}$$

Using $\sin^2(3x) + \cos^2(3x) = 1$,

$$f''(x) = \frac{-9\sin 3x - 9}{(1 + \sin 3x)^2}$$

$$= \frac{-9\,(1+\sin 3x)}{(1+\sin 3x)^2}$$

$$= \frac{-9}{1+\sin 3x} \quad \text{(shown)}$$

To check your answer: at $x = 0$, $f''(0) = \frac{-9}{1+\sin 0} = -9$, matching direct numerical differentiation of $f(x)$ twice at $x = 0$. ✓

$\therefore k = -9$.

Part (ii) 4 marks

APPROACH

Since $f(0) = 0$, the constant term of the Maclaurin series vanishes, so the first non-zero term is $f'(0)x$. We need the third derivative to reach the third non-zero term, found by differentiating the compact form of $f''(x)$ from part (i) rather than differentiating $f'(x)$ directly again.

We evaluate $f(x)$ and its derivatives at $x = 0$.

$$f(0) = \ln(1+\sin 0) = \ln 1 = 0$$

$$f'(0) = \frac{3\cos 0}{1+\sin 0} = 3$$

$$f''(0) = \frac{-9}{1+\sin 0} = -9$$

Differentiating $f''(x) = -9\,(1+\sin 3x)^{-1}$ using the chain rule,

$$f'''(x) = -9 \times (-1)\,(1+\sin 3x)^{-2} \times 3\cos 3x$$

$$= \frac{27\cos 3x}{(1+\sin 3x)^2}$$

$$f'''(0) = \frac{27\cos 0}{(1+\sin 0)^2} = 27$$

The Maclaurin series is $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$.

$$f(x) = 0 + 3x + \frac{-9}{2}x^2 + \frac{27}{6}x^3 + \cdots$$

$$= 3x - \frac{9}{2}x^2 + \frac{9}{2}x^3 + \cdots$$

To check your answer: a computer-algebra Maclaurin expansion of $\ln(1+\sin 3x)$ up to x^3 confirms $3x - \frac{9}{2}x^2 + \frac{9}{2}x^3$. ✓

$\therefore$ the first three non-zero terms are $3x - \frac{9}{2}x^2 + \frac{9}{2}x^3$.

Question 4 7 marks

Complex numbers · polar form · arguments

Part (i) 4 marks

APPROACH

Since we may not use a calculator, we convert each of z_1, z_2, z_3 to modulus-argument form and combine using $\left|\frac{z_1}{z_2 z_3}\right| = \frac{|z_1|}{|z_2||z_3|}$ and $\arg\left(\frac{z_1}{z_2 z_3}\right) = \arg z_1 - \arg z_2 - \arg z_3$, rather than multiplying the complex numbers out in cartesian form.

For $z_1 = 1 + \sqrt{3}i$, the point $(1, \sqrt{3})$ lies in the first quadrant, so

$$|z_1| = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2, \;\; \arg(z_1) = \arctan\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$$

For $z_2 = 1 - i$, the point $(1, -1)$ lies in the fourth quadrant, so

$$|z_2| = \sqrt{1^2 + (-1)^2} = \sqrt{2}, \;\; \arg(z_2) = -\frac{\pi}{4}$$

For $z_3 = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$, reading directly from the given polar form,

$$|z_3| = 2, \;\; \arg(z_3) = \frac{\pi}{6}$$

For the quotient, moduli divide and arguments subtract.

$$\begin{aligned}\left|\frac{z_1}{z_2 z_3}\right| &= \frac{|z_1|}{|z_2|\,|z_3|} \\ &= \frac{2}{\sqrt{2} \times 2} \\ &= \frac{1}{\sqrt{2}} \\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

$$\begin{aligned}\arg\left(\frac{z_1}{z_2 z_3}\right) &= \arg(z_1) - \arg(z_2) - \arg(z_3) \\ &= \frac{\pi}{3} - \left(-\frac{\pi}{4}\right) - \frac{\pi}{6} \\ &= \frac{4\pi}{12} + \frac{3\pi}{12} - \frac{2\pi}{12} \\ &= \frac{5\pi}{12}\end{aligned}$$

Since $\frac{5\pi}{12}$ already lies in $(-\pi, \pi]$, no adjustment is needed.

To check your answer: $\frac{5\pi}{12} = 75°$ lies between $\frac{\pi}{3} = 60°$ and $\frac{\pi}{3} + \frac{\pi}{4} = 105°$, the two extremes before subtracting $\frac{\pi}{6}$. ✓

$\therefore \frac{z_1}{z_2 z_3} = \frac{\sqrt{2}}{2}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$.

Part (ii) 3 marks

> **APPROACH**
>
> A complex number with modulus 1 that is purely imaginary must be i or $-i$. We set $\frac{z_1 z_4}{z_2 z_3}$ equal to each in turn and divide by the result of part (i) to isolate z_4.

Since $\left|\frac{z_1 z_4}{z_2 z_3}\right| = 1$ and $\frac{z_1 z_4}{z_2 z_3}$ is purely imaginary, it equals either $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$ or $-i = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)$.

Writing $\frac{z_1 z_4}{z_2 z_3} = \frac{z_1}{z_2 z_3} \times z_4$ and using part (i),

$$z_4 = \frac{z_1 z_4 / (z_2 z_3)}{z_1 / (z_2 z_3)}$$

Case 1: $\frac{z_1 z_4}{z_2 z_3} = i = 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$.

$$|z_4| = \frac{1}{\sqrt{2}/2} = \sqrt{2}$$

$$\arg(z_4) = \frac{\pi}{2} - \frac{5\pi}{12} = \frac{6\pi}{12} - \frac{5\pi}{12} = \frac{\pi}{12}$$

Case 2: $\frac{z_1 z_4}{z_2 z_3} = -i = 1\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$.

$$|z_4| = \sqrt{2}$$

$$\arg(z_4) = -\frac{\pi}{2} - \frac{5\pi}{12} = -\frac{6\pi}{12} - \frac{5\pi}{12} = -\frac{11\pi}{12}$$

Since $-\frac{11\pi}{12}$ already lies in $(-\pi, \pi]$, no adjustment is needed.

> **To check your answer:** for Case 1, $\frac{5\pi}{12} + \frac{\pi}{12} = \frac{\pi}{2}$, the required argument of $\frac{z_1 z_4}{z_2 z_3}$; for Case 2, $\frac{5\pi}{12} + \left(-\frac{11\pi}{12}\right) = -\frac{\pi}{2}$. ✓

$\therefore z_4 = \sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$ or $z_4 = \sqrt{2}\left(\cos\left(-\frac{11\pi}{12}\right) + i\sin\left(-\frac{11\pi}{12}\right)\right)$.

Question 5 9 marks

Vectors · vector product · area

Part (a) 2 marks

Since the cross product is anti-commutative, $\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b})$ for any vectors $\mathbf{a}$ and $\mathbf{b}$. Substituting the given identity $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{a}$:

$$\mathbf{a} \times \mathbf{b} = -(\mathbf{a} \times \mathbf{b})$$

$$\Rightarrow 2(\mathbf{a} \times \mathbf{b}) = \mathbf{0}$$

$$\Rightarrow \mathbf{a} \times \mathbf{b} = \mathbf{0}$$

Since $\mathbf{a}$ and $\mathbf{b}$ are both non-zero, $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ means $\mathbf{a}$ and $\mathbf{b}$ must be parallel.

> **To check your answer:** if $\mathbf{a}$ and $\mathbf{b}$ are parallel, say $\mathbf{b} = k\mathbf{a}$ for some scalar k, then $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times (k\mathbf{a}) = k(\mathbf{a} \times \mathbf{a}) = \mathbf{0}$, and likewise $\mathbf{b} \times \mathbf{a} = \mathbf{0}$, so indeed $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{a}$. ✓

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$\therefore$ $\mathbf{a}$ and $\mathbf{b}$ are parallel.

Part (b)(i) 3 marks

Since $\mathbf{q}$ is non-zero, the condition $(\mathbf{r}-\mathbf{p})\times\mathbf{q}=\mathbf{0}$ means, by the same reasoning as part (a), that $\overrightarrow{PR}=\mathbf{r}-\mathbf{p}$ is parallel to $\mathbf{q}$.

$\therefore$ R lies on the straight line through P with direction vector $\mathbf{q}$.

Part (b)(ii) 4 marks

Using the given position vectors,

$$\mathbf{r}-\mathbf{p}=\begin{pmatrix}x-(-1)\\y-2\\z-4\end{pmatrix}=\begin{pmatrix}x+1\\y-2\\z-4\end{pmatrix}$$

Forming the dot product with $\mathbf{q}$,

$$\begin{aligned}&(\mathbf{r}-\mathbf{p})\cdot\mathbf{q}\\&=3(x+1)-5(y-2)+2(z-4)\\&=3x+3-5y+10+2z-8\\&=3x-5y+2z+5\end{aligned}$$

Setting this equal to 0,

$$\begin{aligned}&3x-5y+2z+5=0\\&\Rightarrow 3x-5y+2z=-5\end{aligned}$$

To check your answer: substituting $P=(-1,2,4)$ into the equation, $3(-1)-5(2)+2(4)=-3-10+8=-5$, matching the right side — as expected, since a plane through P must contain P. ✓

Since $(\mathbf{r}-\mathbf{p})\cdot\mathbf{q}=0$ means $\overrightarrow{PR}$ is perpendicular to $\mathbf{q}$, the point R lies on the plane through P with normal vector $\mathbf{q}$.

$\therefore$ $3x-5y+2z=-5$; R lies on the plane through P with normal vector $\mathbf{q}$.

Question 6 8 marks

Complex numbers · roots of quadratics

APPROACH

The equation has one real coefficient (t) and two non-real ones, so a single complex equation in z becomes, once we substitute the known root $z=k+ki$, two real equations — real part $=0$ and imaginary part $=0$ — which we solve for k and t together.

Let $z=k+ki=k(1+i)$. We first compute z^2:

$$z^2 = k^2(1+i)^2$$
$$= k^2\left(1+2i+i^2\right)$$
$$= k^2(1+2i-1)$$
$$= 2k^2 i$$

Then $z^2(2+i)$:

$$z^2(2+i) = 2k^2 i(2+i)$$
$$= 4k^2 i + 2k^2 i^2$$
$$= 4k^2 i - 2k^2$$

And $-8iz$:

$$-8iz = -8i \cdot k(1+i)$$
$$= -8ki - 8ki^2$$
$$= -8ki + 8k$$

Substituting all three into $z^2(2+i) - 8iz + t = 0$ and grouping real and imaginary parts,

$$\left(4k^2 i - 2k^2\right) + (-8ki + 8k) + t = 0$$
$$\Rightarrow \left(-2k^2 + 8k + t\right) + \left(4k^2 - 8k\right) i = 0$$

Since k and t are real, both the real part and the imaginary part must vanish. From the imaginary part:

$$4k^2 - 8k = 0$$
$$\Rightarrow 4k(k-2) = 0$$
$$\Rightarrow k = 0 \ \text{(rej as k is non-zero)}, \ k = 2$$

Substituting $k = 2$ into the real part,

$$-2(2)^2 + 8(2) + t = 0$$
$$\Rightarrow -8 + 16 + t = 0$$
$$\Rightarrow t = -8$$

To check your answer: with $k = 2$, $z = 2 + 2i$; substituting into $z^2(2+i) - 8iz + t = 0$ with $t = -8$: $z^2 = (2+2i)^2 = 8i$, so $8i(2+i) - 8i(2+2i) - 8 = (16i - 8) - (16i - 16) - 8 = 0$. ✓

$\therefore t = -8$, $k = 2$, and one root is $z = 2 + 2i$.

To find the other root, we use that for a quadratic $az^2 + bz + c = 0$, the sum of roots is $-\frac{b}{a}$. Here $a = 2 + i$, $b = -8i$, $c = t = -8$:

$$z_1 + z_2 = -\frac{b}{a} = \frac{8i}{2+i}$$

Rationalising by multiplying by $\frac{2-i}{2-i}$,

$$\frac{8i}{2+i}$$
$$= \frac{8i(2-i)}{(2+i)(2-i)}$$
$$= \frac{16i - 8i^2}{5}$$

$$= \frac{16i + 8}{5}$$
$$= \frac{8}{5} + \frac{16}{5}i$$

Since $z_1 = 2 + 2i$,

$$z_2 = \left(\frac{8}{5} + \frac{16}{5}i\right) - (2 + 2i)$$
$$= \left(\frac{8}{5} - \frac{10}{5}\right) + \left(\frac{16}{5} - \frac{10}{5}\right)i$$
$$= -\frac{2}{5} + \frac{6}{5}i$$

To check your answer: solving $z^2(2+i) - 8iz - 8 = 0$ directly confirms the two roots are exactly $2 + 2i$ and $-\frac{2}{5} + \frac{6}{5}i$. ✓

$\therefore$ the other root is $z = -\frac{2}{5} + \frac{6}{5}i$.

Question 7 9 marks

Integration · by parts · double angles

Part (i) 1 mark

$$\int f(x)\,dx = 2x + \frac{1}{4}\cos 4x + C$$

Explanation: integrating term by term, $\int 2\,dx = 2x$, and $\int \sin 4x\,dx = -\frac{1}{4}\cos 4x$ by the reverse chain rule (dividing by the derivative of the angle $4x$). Hence $\int (2 - \sin 4x)\,dx = 2x - \left(-\frac{1}{4}\cos 4x\right) + C = 2x + \frac{1}{4}\cos 4x + C$.

To check your answer: differentiating $2x + \frac{1}{4}\cos 4x + C$ gives $2 - \frac{1}{4}(4\sin 4x) = 2 - \sin 4x = f(x)$, matching the original integrand. ✓

$\therefore \int f(x)\,dx = 2x + \frac{1}{4}\cos 4x + C$.

Part (ii) 4 marks

APPROACH

$xf(x) = 2x - x\sin 4x$ splits into an easy polynomial integral and a product $x\sin 4x$, which needs integration by parts (a polynomial times a sine cannot be integrated directly).

$$\int_0^{\frac{1}{2}\pi} xf(x)\,dx = \int_0^{\frac{1}{2}\pi} 2x\,dx - \int_0^{\frac{1}{2}\pi} x\sin 4x\,dx$$

The first integral:

$$\int_0^{\frac{1}{2}\pi} 2x\,dx = \left[x^2\right]_0^{\frac{1}{2}\pi} = \left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{4}$$

For the second, we use integration by parts with $u = x$ (so $du = dx$) and $dv = \sin 4x\,dx$ (so $v = -\frac{1}{4}\cos 4x$):

$$\begin{aligned}\int x\sin 4x\,dx &= -\frac{x}{4}\cos 4x + \int \frac{1}{4}\cos 4x\,dx \\ &= -\frac{x}{4}\cos 4x + \frac{1}{16}\sin 4x\end{aligned}$$

Evaluating from 0 to $\frac{\pi}{2}$,

$$\begin{aligned}\left[-\frac{x}{4}\cos 4x + \frac{1}{16}\sin 4x\right]_0^{\frac{\pi}{2}} &= \left(-\frac{\pi/2}{4}\cos 2\pi + \frac{1}{16}\sin 2\pi\right) - (0+0) \\ &= -\frac{\pi}{8}(1) + 0 - 0 \\ &= -\frac{\pi}{8}\end{aligned}$$

Combining the two integrals,

$$\int_0^{\frac{1}{2}\pi} xf(x)\,dx = \frac{\pi^2}{4} - \left(-\frac{\pi}{8}\right) = \frac{\pi^2}{4} + \frac{\pi}{8}$$

To check your answer: numerical integration of $x\,(2-\sin 4x)$ from 0 to $\frac{\pi}{2}$ gives ≈ 2.860, and $\frac{\pi^2}{4} + \frac{\pi}{8} \approx 2.467 + 0.393 \approx 2.860$. ✓

$\therefore \int_0^{\frac{1}{2}\pi} xf(x)\,dx = \frac{\pi^2}{4} + \frac{\pi}{8}$.

Part (iii) 4 marks

APPROACH

$(f(x))^2$ contains $\sin^2(4x)$, which is not directly integrable; the double angle formula $\sin^2(4x) = \frac{1-\cos 8x}{2}$ turns it into a sum of integrable terms.

$$(f(x))^2 = (2-\sin 4x)^2 = 4 - 4\sin 4x + \sin^2(4x)$$

Using $\sin^2(4x) = \frac{1-\cos 8x}{2}$,

$$\begin{aligned}&(f(x))^2 \\ &= 4 - 4\sin 4x + \frac{1}{2} - \frac{1}{2}\cos 8x \\ &= \frac{9}{2} - 4\sin 4x - \frac{1}{2}\cos 8x\end{aligned}$$

Integrating term by term,

$$\int_0^{\frac{1}{2}\pi} (f(x))^2\,dx = \left[\frac{9}{2}x + \cos 4x - \frac{1}{16}\sin 8x\right]_0^{\frac{\pi}{2}}$$

At $x = \frac{\pi}{2}$:

$$\frac{9}{2}\left(\frac{\pi}{2}\right) + \cos 2\pi - \frac{1}{16}\sin 4\pi = \frac{9\pi}{4} + 1 - 0 = \frac{9\pi}{4} + 1$$

At $x = 0$:

$$0 + \cos 0 - \frac{1}{16}\sin 0 = 0 + 1 - 0 = 1$$

Subtracting,

$$\int_0^{\frac{1}{2}\pi} (f(x))^2\,dx = \left(\frac{9\pi}{4} + 1\right) - 1 = \frac{9\pi}{4}$$

To check your answer: numerical integration of $(2 - \sin 4x)^2$ from 0 to $\frac{\pi}{2}$ gives ≈ 7.069, and $\frac{9\pi}{4} \approx 7.069$. ✓

$\therefore \int_0^{\frac{1}{2}\pi} (f(x))^2\,dx = \frac{9\pi}{4}$.

Question 8 12 marks

Sequences and series · AP and GP

Part (a)(i) 2 marks

Let the first term be $a = 4$ and the common difference d. Since the 5th term is 10,

$$\begin{aligned} a + 4d &= 10 \\ \Rightarrow 4 + 4d &= 10 \\ \Rightarrow 4d &= 6 \\ \Rightarrow d &= 1.5 \end{aligned}$$

The 30th term is

$$a + 29d = 4 + 29(1.5) = 4 + 43.5 = 47.5$$

To check your answer: the 5th term is $4 + 4(1.5) = 4 + 6 = 10$, matching the given value. ✓

$\therefore$ the 30th term is 47.5.

Part (a)(ii) 3 marks

The sum of the first n terms of an AP is $S_n = \frac{n}{2}(2a + (n-1)d)$. The sum from the 21st to the 50th term inclusive is $S_{50} - S_{20}$:

$$S_{50} = \frac{50}{2}(2(4) + 49(1.5)) = 25(8 + 73.5) = 25(81.5) = 2037.5$$

$$S_{20} = \frac{20}{2}(2(4) + 19(1.5)) = 10(8 + 28.5) = 10(36.5) = 365$$

$$S_{50} - S_{20} = 2037.5 - 365 = 1672.5$$

To check your answer: the 21st and 50th terms are $4 + 20(1.5) = 34$ and $4 + 49(1.5) = 77.5$; summing these 30 terms as $\frac{30}{2}(34 + 77.5) = 15(111.5) = 1672.5$ matches the difference of sums above. ✓

$\therefore$ the sum from the 21st to the 50th term inclusive is 1672.5.

Part (b)(i) 2 marks

Let the first term be $a = 4$ and common ratio $r > 0$. Since the 5th term is 1.6384,

$$ar^4 = 1.6384$$
$$\Rightarrow 4r^4 = 1.6384$$
$$\Rightarrow r^4 = 0.4096$$

Recognising $0.4096 = 0.8^4$ (since $0.8^2 = 0.64$ and $0.64^2 = 0.4096$), and taking the positive root,

$$r = 0.8$$

Since $|r| = 0.8 < 1$, the sum to infinity exists:

$$S_\infty = \frac{a}{1-r} = \frac{4}{1-0.8} = \frac{4}{0.2} = 20$$

To check your answer: the 5th term is $4(0.8)^4 = 4(0.4096) = 1.6384$, matching the given value. ✓

$\therefore S_\infty = 20$.

Part (b)(ii) 5 marks

APPROACH

We first write S_n in terms of $r = 0.8$ and manipulate the given inequality $S_n > 19.6$ algebraically to reach $0.8^n < 0.02$, tracking the direction of each inequality carefully — multiplying by -1 reverses it. Taking logarithms then isolates n; since $\ln 0.8 < 0$, dividing by it reverses the inequality again.

The sum of the first n terms is

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{4(1-0.8^n)}{0.2} = 20(1-0.8^n)$$

Given $S_n > 19.6$,

$$20(1-0.8^n) > 19.6$$
$$\Rightarrow 1 - 0.8^n > 0.98$$
$$\Rightarrow -0.8^n > -0.02$$
$$\Rightarrow 0.8^n < 0.02 \ \ (\text{multiplying by } -1, \text{reversing the inequality})$$

(shown)

Taking logarithms of both sides (noting $\ln 0.8 < 0$, so the inequality reverses),

$$0.8^n < 0.02$$
$$\Rightarrow n \ln 0.8 < \ln 0.02$$
$$\Rightarrow n > \frac{\ln 0.02}{\ln 0.8}$$

To check your answer: $\frac{\ln 0.02}{\ln 0.8} \approx \frac{-3.912}{-0.2231} \approx 17.53$; testing $n = 17$: $S_{17} = 20\left(1 - 0.8^{17}\right) \approx 19.55 < 19.6$ (fails), and $n = 18$: $S_{18} = 20\left(1 - 0.8^{18}\right) \approx 19.64 > 19.6$ (succeeds). ✓

Since n must be an integer and $n > 17.53$, the smallest possible value is $n = 18$.

$\therefore$ the smallest possible value of n is 18.

Question 9 12 marks

Integration · intersecting curves · area

Part (i) 1 mark

Let l_1 be the line through the origin with gradient 2, making an angle $\tan^{-1}(2)$ with the positive x-axis, and l_2 the line through the origin with gradient $-\frac{1}{2}$, making an angle $\tan^{-1}\left(-\frac{1}{2}\right)$ with the positive x-axis. We check the product of the two gradients.

$$\begin{aligned} & 2 \times \left(-\frac{1}{2}\right) \\ &= -1 \end{aligned}$$

Since the product of the gradients is -1, l_1 and l_2 are perpendicular, so the angle between them is $\frac{1}{2}\pi$. This angle is exactly the difference between the two inclinations, measured from l_2 up to l_1:

$$\tan^{-1}(2) - \tan^{-1}\left(-\frac{1}{2}\right) = \frac{1}{2}\pi \quad \text{(shown)}$$

Part (ii) 3 marks

APPROACH

"C_1 and C_2 intersect" means the equation $C_1 = C_2$ has at least one real solution, so we equate the two curves, reduce to a quadratic in x, and demand a non-negative discriminant.

Setting C_1 and C_2 equal,

$$\frac{1}{x^2 + 1} = \frac{k}{3x + 4}$$

Cross-multiplying,

$$\begin{aligned} 3x + 4 &= k\left(x^2 + 1\right) \\ \Rightarrow kx^2 - 3x + (k - 4) &= 0 \end{aligned}$$

Since $k > 0$, this is a genuine quadratic in x, so real intersection requires its discriminant to be non-negative.

$$\begin{aligned} (-3)^2 - 4k(k - 4) &\geq 0 \\ \Rightarrow 9 - 4k^2 + 16k &\geq 0 \\ \Rightarrow 4k^2 - 16k - 9 &\leq 0 \end{aligned}$$

Factorising the quadratic in k,

$$4k^2 - 16k - 9 = (2k - 9)(2k + 1)$$

so

$$(2k - 9)(2k + 1) \leq 0$$
$$\Rightarrow -\frac{1}{2} \leq k \leq \frac{9}{2}$$

Combined with the given condition $k > 0$,

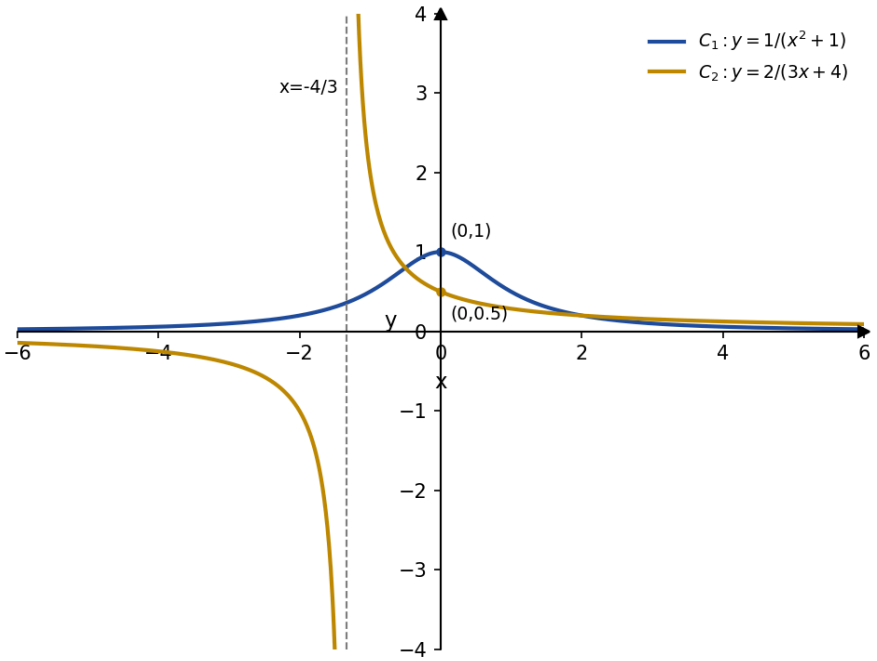
$$\therefore\ 0 < k \leq \frac{9}{2}$$

To check your answer: at $k = \frac{9}{2}$, the discriminant is $9 - 4\left(\frac{9}{2}\right)\left(\frac{9}{2} - 4\right) = 9 - 4\left(\frac{9}{2}\right)\left(\frac{1}{2}\right) = 9 - 9 = 0$, confirming a single (repeated) intersection at this boundary. ✓

Part (iii) 3 marks

With $k = 2$: $C_1 : y = \frac{1}{x^2+1}$ has y-intercept $(0, 1)$ (its maximum, since C_1 decreases as $|x|$ increases), no x-intercept, and horizontal asymptote $y = 0$.

$C_2 : y = \frac{2}{3x+4}$ has y-intercept $\left(0, \frac{1}{2}\right)$, no x-intercept, a vertical asymptote $x = -\frac{4}{3}$, and horizontal asymptote $y = 0$.



$\therefore C_1$ has y-intercept $(0, 1)$ and asymptote $y = 0$; C_2 has y-intercept $\left(0, \frac{1}{2}\right)$ and asymptotes $x = -\frac{4}{3}$, $y = 0$.

Part (iv) 5 marks

APPROACH

the bounded area needs the exact intersections of C_1 and C_2 at $k = 2$, then an integral of the difference; the result is arranged to reuse the identity from part (i).

With $k = 2$, intersections satisfy $2x^2 - 3x - 2 = 0$.

$$2x^2 - 3x - 2 = 0$$
$$\Rightarrow (2x + 1)(x - 2) = 0$$
$$\Rightarrow x = -\frac{1}{2} \text{ or } x = 2$$

From the sketch in part (iii), C_1 lies above C_2 on $\left[-\frac{1}{2}, 2\right]$, so

$$Area = \int_{-1/2}^{2} \left(\frac{1}{x^2 + 1} - \frac{2}{3x + 4} \right) dx$$

Using $\int \frac{1}{x^2+1}\, dx = \tan^{-1} x$ and $\int \frac{2}{3x+4}\, dx = \frac{2}{3} \ln |3x + 4|$,

$$Area = \left[\tan^{-1} x - \frac{2}{3} \ln |3x + 4| \right]_{-1/2}^{2}$$
$$= \left(\tan^{-1}(2) - \frac{2}{3} \ln 10 \right) - \left(\tan^{-1}\left(-\frac{1}{2}\right) - \frac{2}{3} \ln \frac{5}{2} \right)$$
$$= \left[\tan^{-1}(2) - \tan^{-1}\left(-\frac{1}{2}\right) \right] - \frac{2}{3} \left(\ln 10 - \ln \frac{5}{2} \right)$$

Using the result of part (i), $\tan^{-1}(2) - \tan^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{2}$, and $\ln 10 - \ln \frac{5}{2} = \ln 4 = 2 \ln 2$,

$$Area = \frac{\pi}{2} - \frac{2}{3} (2 \ln 2)$$
$$= \frac{\pi}{2} - \frac{4}{3} \ln 2$$

To check your answer: $\frac{\pi}{2} - \frac{4}{3} \ln 2 \approx 0.647$; numerically integrating $\frac{1}{x^2+1} - \frac{2}{3x+4}$ from $-\frac{1}{2}$ to 2 also gives ≈ 0.647. ✓

$\therefore$ Area $= \frac{\pi}{2} - \frac{4}{3} \ln 2 \approx 0.647$ units^2.

Question 10 14 marks

Differential equations · population model

Part (i) 2 marks

Since the death rate exceeds the birth rate by 3% of the population every year, P decreases at a rate proportional to itself, with constant of proportionality 0.03.

$$\frac{dP}{dt} = -0.03P$$

Part (ii) 4 marks

Separating variables and integrating,

$$\int \frac{1}{P}\, dP = \int -0.03\, dt$$
$$\Rightarrow \ln P = -0.03t + c$$

$$\Rightarrow P = Ae^{-0.03t}, \text{ where } A = e^c$$

Let P_0 be the initial number of sheep, at $t = 0$. Then $A = P_0$:

$$P = P_0 e^{-0.03t}$$

To check your answer: differentiating gives $\frac{dP}{dt} = -0.03P_0 e^{-0.03t} = -0.03P$, matching the differential equation of part (i). ✓

As $t \to \infty$, $e^{-0.03t} \to 0$, so $P \to 0$.

$\therefore P = P_0 e^{-0.03t}$; if this situation continues over many years, the sheep population decreases indefinitely towards zero, i.e. it eventually dies out.

Part (iii) 2 marks

Sheep are now imported at a constant rate of n per year, added to the existing 3% decline.

$$\frac{dP}{dt} = -0.03P + n$$

Part (iv) 4 marks

APPROACH

with a constant term added, we still separate variables, but the left side integrates to a logarithm of $n - 0.03P$ rather than of P alone.

$$\int \frac{1}{n - 0.03P}\, dP = \int dt$$

$$\Rightarrow -\frac{1}{0.03} \ln|n - 0.03P| = t + c$$

$$\Rightarrow \ln|n - 0.03P| = -0.03t - 0.03c$$

$$\Rightarrow n - 0.03P = Be^{-0.03t}, \text{ where } B = \pm e^{-0.03c}$$

$$P = \frac{n - Be^{-0.03t}}{0.03}$$

$$= \frac{n}{0.03} - \frac{B}{0.03} e^{-0.03t}$$

Using the initial condition $P = P_0$ at $t = 0$,

$$P_0 = \frac{n}{0.03} - \frac{B}{0.03}$$

$$\Rightarrow \frac{B}{0.03} = \frac{n}{0.03} - P_0$$

Substituting back,

$$P = \frac{n}{0.03} - \left(\frac{n}{0.03} - P_0\right) e^{-0.03t}$$

$$= \frac{n}{0.03} + \left(P_0 - \frac{n}{0.03}\right) e^{-0.03t}$$

To check your answer: at $t = 0$, $P = \frac{n}{0.03} + \left(P_0 - \frac{n}{0.03}\right) = P_0$ ✓; differentiating gives $\frac{dP}{dt} = -0.03\left(P_0 - \frac{n}{0.03}\right)e^{-0.03t} = -0.03\left(P - \frac{n}{0.03}\right) = -0.03P + n$, matching the differential equation of part (iii). ✓

$\therefore P = \frac{n}{0.03} + \left(P_0 - \frac{n}{0.03}\right)e^{-0.03t}$.

Part (v) 2 marks

As $t \to \infty$, $e^{-0.03t} \to 0$, so P settles to the steady-state value $\frac{n}{0.03}$. Setting this equal to 500,

$$\frac{n}{0.03} = 500$$
$$\Rightarrow n = 500 \times 0.03$$
$$= 15$$

To check your answer: with $n = 15$, the steady-state population is $\frac{15}{0.03} = 500$, matching the given value. ✓

$\therefore n = 15$.

Question 11 12 marks

Differentiation · maximising an angle

Part (i) 3 marks

APPROACH

$\theta = \angle\mathrm{CKD}$ is not the base angle of a single right-angled triangle, but it is the difference of two angles that are: $\angle\mathrm{AKD}$ and $\angle\mathrm{AKC}$, each in a triangle right-angled at A.

Since $AB \perp XY$, triangles AKC and AKD are right-angled at A. Let $\alpha = \angle\mathrm{AKC}$ and $\beta = \angle\mathrm{AKD}$. With $AC = a$ and $AK = x$,

$$\tan\alpha = \frac{AC}{AK} = \frac{a}{x}$$

Since $AD = AC + CD = a + 4$,

$$\tan\beta = \frac{AD}{AK} = \frac{a+4}{x}$$

From the diagram, $\theta = \angle\mathrm{CKD} = \beta - \alpha$. Using the tangent subtraction formula,

$$\tan\theta = \tan(\beta - \alpha)$$
$$= \frac{\tan\beta - \tan\alpha}{1 + \tan\beta\tan\alpha}$$
$$= \frac{\frac{a+4}{x} - \frac{a}{x}}{1 + \frac{a+4}{x} \cdot \frac{a}{x}}$$

$$= \frac{\frac{4}{x}}{1 + \frac{a(a+4)}{x^2}}$$

Multiplying numerator and denominator by x^2,

$$\tan\theta = \frac{4x}{x^2 + a(a+4)}$$
$$= \frac{4x}{x^2 + a^2 + 4a} \quad \text{(shown)}$$

Part (ii) 3 marks

Treating a as constant, we differentiate $\tan\theta = \frac{4x}{x^2+a^2+4a}$ with respect to x by the quotient rule, writing $c = a^2 + 4a$ for brevity.

$$\frac{d(\tan\theta)}{dx} = \frac{4\left(x^2 + c\right) - 4x(2x)}{\left(x^2 + c\right)^2}$$
$$= \frac{4c - 4x^2}{\left(x^2 + c\right)^2}$$

Setting this to zero for a stationary value,

$$4c - 4x^2 = 0$$
$$\Rightarrow x^2 = c = a^2 + 4a$$
$$\Rightarrow x = \sqrt{a(a+4)}$$

taking the positive root since x is a length. Substituting back,

$$\tan\theta = \frac{4\sqrt{a(a+4)}}{a(a+4) + a(a+4)}$$
$$= \frac{4\sqrt{a(a+4)}}{2a(a+4)}$$
$$= \frac{2}{\sqrt{a(a+4)}}$$

To check your answer: with $a = 3$, $x = \sqrt{21} \approx 4.58$ and $\tan\theta \approx 0.436$; testing nearby $x = 4$ and $x = 5$ in $\tan\theta = 4x/\left(x^2 + 21\right)$ gives 0.432 and 0.435, both smaller, confirming a maximum near $x \approx 4.58$. ✓

$\therefore x = \sqrt{a(a+4)}$, giving $\tan\theta_{\max} = \frac{2}{\sqrt{a(a+4)}}$.

Part (iii) 1 mark

The optimal point gives the largest angle θ, but the corresponding kicking distance $AK = x = \sqrt{a(a+4)}$ may be too far from the posts for the player to strike the ball with enough power or accuracy.

$\therefore$ A player may prefer a closer point with a slightly smaller angle but a shorter, more reliable kick.

Part (iv) 3 marks

Triangle AKD is right-angled at A, so

$$\tan(\angle \mathrm{KDA}) = \frac{AK}{AD} = \frac{x}{a+4}$$

At the optimal point, $x = \sqrt{a(a+4)}$, so

$$\begin{aligned}\tan(\angle \mathrm{KDA}) &= \frac{\sqrt{a(a+4)}}{a+4} \\ &= \frac{\sqrt{a}\sqrt{a+4}}{a+4} \\ &= \sqrt{\frac{a}{a+4}} \quad \text{(shown)}\end{aligned}$$

To check your answer: with $a = 3$, $\tan(\angle \mathrm{KDA}) = \sqrt{3/7} \approx 0.655$, and directly $x/(a+4) = \sqrt{21}/7 \approx 0.655$, matching. ✓

When a is much greater than 4, $\frac{a}{a+4} \to 1$, so $\tan(\angle \mathrm{KDA}) \to 1$.

$\therefore \angle \mathrm{KDA} \to 45^\circ$ (i.e. $\frac{\pi}{4}$) when $a \gg 4$.

Part (v) 2 marks

Since $XC = DY$ and $XY = XC + CD + DY = 50$,

$$\begin{aligned}2(XC) + 4 &= 50 \\ \Rightarrow XC &= 23\end{aligned}$$

As A varies between X and C, $a = AC$ ranges strictly between 0 (at $A = C$) and 23 (at $A = X$).

From part (ii), the optimal angle satisfies $\tan\theta = \frac{2}{\sqrt{a(a+4)}}$, which decreases as a increases (since $a(a+4)$ increases). So θ is largest when a is smallest and smallest when a is largest.

As $a \to 0^+$ (i.e. $A \to C$), $\tan\theta \to \infty$, so $\theta \to 90^\circ$.

As $a \to 23^-$ (i.e. $A \to X$),

$$\tan\theta \to \frac{2}{\sqrt{23(27)}} = \frac{2}{\sqrt{621}} \approx 0.0803$$

giving $\theta \approx 4.59^\circ$.

To check your answer: at $a = 10$ (between the two extremes), $\tan\theta = 2/\sqrt{140} \approx 0.169$, giving $\theta \approx 9.59^\circ$ — larger than at $a = 23$, consistent with θ decreasing as a increases. ✓

$\therefore 4.59^\circ < \theta < 90^\circ$ (3 s.f.), as A varies between X and C.