TEN-YEAR SERIES · WORKED SOLUTIONS

2019 A-Level H2 Mathematics Paper 1

12 questions · 100 marks

The question paper is © UCLES/SEAB and is not reproduced here — have your own copy of the paper open as you work through these solutions. The worked solutions and diagrams are original work by The Learning Cave and are for personal study only.

thelearningcave.com.sg

Physics. Math. Chemistry.

Question 1 4 marks

Complex numbers · conjugate roots

APPROACH

f has real coefficients, so non-real roots occur in conjugate pairs. Since $2+i$ is a root, $2-i$ must also be a root. Together with the given root -3, this accounts for all three roots of the cubic, and $f(z)$ factorises completely in terms of a.

Since $2+i$, $2-i$ and -3 are the three roots of $f(z)=0$,

$$f(z) = a\,(z-(2+i))\,(z-(2-i))\,(z+3)$$

Multiplying the conjugate pair first,

$$\begin{aligned}&(z-(2+i))\,(z-(2-i))\\ &= ((z-2)-i)\,((z-2)+i)\\ &= (z-2)^2+1\\ &= z^2-4z+5\end{aligned}$$

So

$$\begin{aligned}f(z) &= a\left(z^2-4z+5\right)(z+3)\\ &= a\left(z^3+3z^2-4z^2-12z+5z+15\right)\\ &= a\left(z^3-z^2-7z+15\right)\\ &= az^3-az^2-7az+15a\end{aligned}$$

Comparing with $f(z)=az^3+bz^2+cz+d$,

$$b=-a,\quad c=-7a,\quad d=15a$$

To check your answer: substituting $z=-3$ into $az^3-az^2-7az+15a$ gives $-27a-9a+21a+15a=0$. ✓

$\therefore b=-a,\ c=-7a,\ d=15a.$

Question 2 5 marks

Integration · area under a curve

Part (i) 1 mark

Since C crosses the x-axis at $(a,0)$, a satisfies

$$x^3+x-1=0$$

By GC, this cubic has only one real root.

$$\Rightarrow a=0.682\ \ (3\text{ d.p.})$$

$\therefore a = 0.682.$

Part (ii) 4 marks

APPROACH

We need to know, on each interval, whether C lies above or below the x-axis, since that decides whether an integral of y gives the area directly or its negative. Since $y' = 3x^2 + 1 > 0$ for all x, C is strictly increasing throughout, and $a = 0.682$ is its only zero. So $y < 0$ for $x < a$ and $y > 0$ for $x > a$.

On $[-1, 0]$, since $0 < a$, C lies entirely below the x-axis, so P is the region between C and the x-axis there and

$$\begin{aligned}\text{Area of } P &= -\int_{-1}^{0} \left(x^3 + x - 1\right)\, dx \\ &= -\left[\frac{x^4}{4} + \frac{x^2}{2} - x\right]_{-1}^{0} \\ &= -\left(0 - \left(\frac{1}{4} + \frac{1}{2} + 1\right)\right) \\ &= \frac{7}{4}\end{aligned}$$

On $[a, b]$, since $b > a$, C lies entirely above the x-axis, so

$$\text{Area of } Q = \int_{a}^{b} \left(x^3 + x - 1\right)\, dx = \left[\frac{x^4}{4} + \frac{x^2}{2} - x\right]_{a}^{b}$$

Using the given condition that the area of Q is 2 times the area of P,

$$\left[\frac{x^4}{4} + \frac{x^2}{2} - x\right]_{a}^{b} = 2\left(\frac{7}{4}\right) = \frac{7}{2}$$

This is an equation in b alone, since $a = 0.682327\ldots$ is already known. By GC, solving numerically,

$$b = 1.892 \ \ (3 \text{ d.p.})$$

To check your answer: with $a = 0.682327803828$ and $b = 1.892432535925$, numerically integrating $x^3 + x - 1$ from a to b gives $1.750000\ldots = 2 \times \frac{7}{4}$. ✓

$\therefore b = 1.892.$

Question 3 5 marks

Graphs and transformations · completing the cube

Part (i) 2 marks

APPROACH

To complete the cube on $x^3 - 3x^2 + 3x$, we compare with the expansion of $(x-1)^3 = x^3 - 3x^2 + 3x - 1$, then account for the constant term left over.

Factoring out 2 from every term,

$$\begin{aligned} f(x) &= 2x^3 - 6x^2 + 6x - 12 \\ &= 2\left(x^3 - 3x^2 + 3x - 6\right) \end{aligned}$$

Since $(x-1)^3 = x^3 - 3x^2 + 3x - 1$,

$$x^3 - 3x^2 + 3x - 6 = (x-1)^3 - 1 - 6 + 1$$

Note: the constant -1 from $(x-1)^3$ has to be removed before the -6 is added, since $(x-1)^3$ already carries a -1.

$$x^3 - 3x^2 + 3x - 6 = (x-1)^3 - 5$$

Therefore

$$f(x) = 2\{(x-1)^3 - 5\}$$

$\therefore p = 2,\ q = -1,\ r = -5.$

To check your answer: $f(1) = 2(1) - 6(1) + 6(1) - 12 = -10$, and $2\{(1-1)^3 - 5\} = 2(-5) = -10$. ✓

Part (ii) 3 marks

Since $f(x) = 2\{(x-1)^3 - 5\}$, we build up $y = f(x)$ from $y = x^3$ in the order the brackets suggest: first the translation inside the cube, then the translation added to it, then the stretch multiplying the whole bracket.

Starting from $y = x^3$:

1. Translate 1 unit in the positive x-direction, giving $y = (x-1)^3$.
2. Translate 5 units in the negative y-direction, giving $y = (x-1)^3 - 5$.
3. Stretch with scale factor 2 parallel to the y-axis (relative to the x-axis), giving $y = 2\{(x-1)^3 - 5\} = f(x)$.

*Working with the expanded form $y = 2(x-1)^3 - 10$ gives an equivalent sequence: translate 1 unit in the positive x-direction, stretch with scale factor 2 parallel to the y-axis, then translate 10 units in the negative y-direction.

$\therefore$ the sequence is: translate by 1 unit in the positive x-direction, translate by 5 units in the negative y-direction, then stretch with scale factor 2 parallel to the y-axis.

THE LEARNING CAVE thelearningcave.com.sg Physics. Math. Chemistry.

Question 4 6 marks

Inequalities · modulus · exponential graphs

Part (i) 3 marks

APPROACH

Sketch $y = 2^x - 10$ first in our heads, then reflect the part below the x-axis upward to get $y = |2^x - 10|$.

Since $2^x > 0$ for all x, $2^x - 10 \to -10$ as $x \to -\infty$ and $2^x - 10 \to \infty$ as $x \to \infty$, increasing throughout, with a single zero where $2^x = 10$.

y-intercept: at $x = 0$,

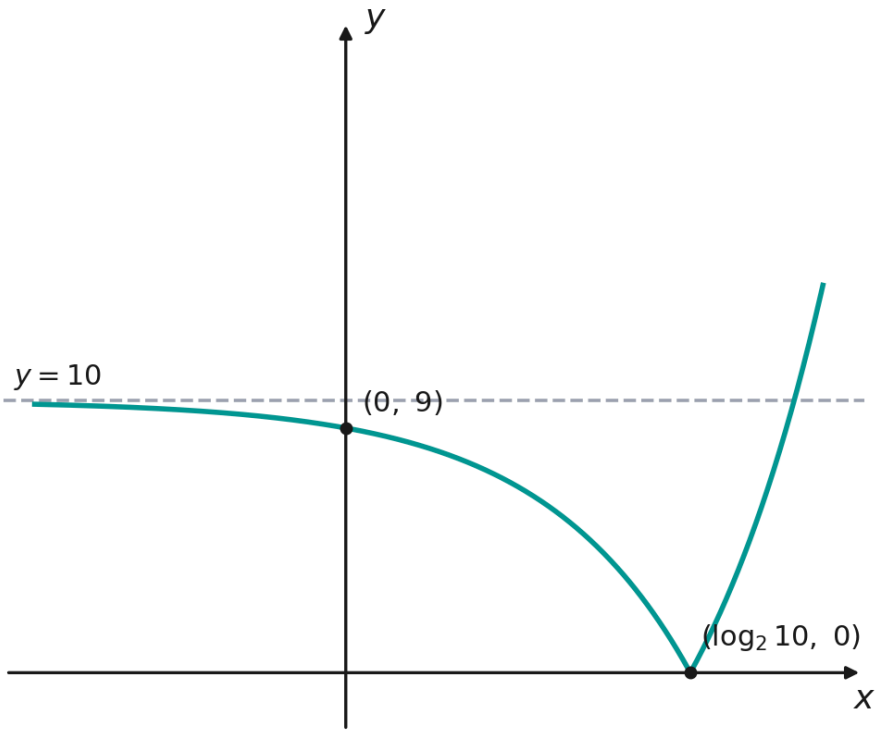
$$y = |2^0 - 10| = |1 - 10| = 9$$

x-intercept: where $2^x - 10 = 0$,

$$2^x = 10$$

$$\Rightarrow x = \log_2 10$$

Since $2^x - 10 \to -10$ as $x \to -\infty$, $|2^x - 10| \to 10$, so $y = 10$ is an asymptote on the left. For $x < \log_2 10$, $2^x - 10 < 0$ so $y = 10 - 2^x$, decreasing from 10 towards 0. For $x > \log_2 10$, $2^x - 10 > 0$ so $y = 2^x - 10$, increasing from 0 without bound.



$\therefore$ the curve meets the axes at $(0,\ 9)$ and $(\log_2 10,\ 0)$, with asymptote $y = 10$.

Part (ii) 3 marks

APPROACH

With the modulus, $|2^x - 10| \leq 6$ is equivalent to $-6 \leq 2^x - 10 \leq 6$. Since 2^x is increasing, taking $\log_2$ across the resulting double inequality for 2^x keeps the direction of both inequalities unchanged.

Since $|2^x - 10| \leq 6$,

$$-6 \leq 2^x - 10 \leq 6$$
$$\Rightarrow 4 \leq 2^x \leq 16$$

Note: 2^x is increasing for all x, so applying $\log_2$ to a double inequality preserves both inequality signs.

$$\Rightarrow \log_2 4 \leq x \leq \log_2 16$$
$$\Rightarrow 2 \leq x \leq 4$$

To check your answer: at $x = 2$, $|2^2 - 10| = |4 - 10| = 6$; at $x = 4$, $|2^4 - 10| = |16 - 10| = 6$. Both endpoints attain equality, and $x = 3$ gives $|8 - 10| = 2 \leq 6$. ✓

$\therefore 2 \leq x \leq 4.$

Question 5 6 marks

Functions · inverse · composite functions

Part (i) 3 marks

Let $y = f(x) = e^{2x} - 4$. To find f^{-1}, we write x in terms of y and then swap the letters.

$$y = e^{2x} - 4$$
$$e^{2x} = y + 4$$
$$2x = \ln(y + 4)$$
$$x = \frac{1}{2}\ln(y + 4)$$

Interchanging x and y,

$$f^{-1}(x) = \frac{1}{2}\ln(x + 4).$$

Note: the domain of f^{-1} is the range of f, not all x for which $\ln(x + 4)$ is defined by itself.

Since $x \in \mathbb{R}$, we have $e^{2x} > 0$ for every x, so

$$f(x) = e^{2x} - 4 > -4,$$

and as $x \to \infty$, $f(x) \to \infty$, while as $x \to -\infty$, $f(x) \to -4^+$. So the range of f is $(-4, \infty)$.

$\therefore f^{-1}(x) = \frac{1}{2}\ln(x + 4)$, with domain $x > -4$.

To check your answer: $f(0) = e^0 - 4 = -3$, and $f^{-1}(-3) = \frac{1}{2}\ln(1) = 0$. ✓

Part (ii) 3 marks

APPROACH

$fg(x)$ means $f\,(g(x))$, so we substitute $g(x) = x + 2$ into f before solving.

Since $g(x) = x + 2$,

$$\begin{aligned} fg(x) &= f(x+2) \\ &= e^{2(x+2)} - 4 \\ &= e^{2x+4} - 4. \end{aligned}$$

Using the given equation $fg(x) = 5$,

$$\begin{aligned} e^{2x+4} - 4 &= 5 \\ e^{2x+4} &= 9 \end{aligned}$$

Taking $\ln$ of both sides,

Since $\ln 9 = \ln 3^2 = 2\ln 3$,

$$2x + 4 = 2\ln 3.$$

Solving:

$$x = \ln 3 - 2$$

$\therefore x = \ln 3 - 2.$

Question 6 6 marks

Sequences and series · method of differences

Part (i) 4 marks

APPROACH

$4r^2 - 1$ factorises as a difference of two squares, $(2r-1)(2r+1)$, so we expect the partial fractions to telescope when summed — each term should cancel with a term two steps away.

Since $4r^2 - 1 = (2r-1)(2r+1)$, we write

$$\frac{1}{4r^2-1} = \frac{A}{2r-1} + \frac{B}{2r+1}$$

for constants A and B. Multiplying through by $(2r-1)(2r+1)$,

$$1 = A(2r+1) + B(2r-1).$$

Substituting $r = \frac{1}{2}$: $1 = 2A$, so $A = \frac{1}{2}$. Substituting $r = -\frac{1}{2}$: $1 = -2B$, so $B = -\frac{1}{2}$.

THE LEARNING CAVE thelearningcave.com.sg Physics. Math. Chemistry.

$$\therefore \frac{1}{4r^2-1} = \frac{1}{2}\left(\frac{1}{2r-1} - \frac{1}{2r+1}\right).$$

To check your answer: at $r = 1$, $\frac{1}{4(1)^2-1} = \frac{1}{3}$, and $\frac{1}{2}\left(1 - \frac{1}{3}\right) = \frac{1}{3}$. ✓

Now we sum, writing out enough terms to see the cancellation:

$$\sum_{r=1}^{n} \frac{1}{4r^2-1} = \frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{2r-1} - \frac{1}{2r+1}\right)$$
$$= \frac{1}{2}\left\{\left(1 - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)\right\}$$

Every term except the first and the last cancels, leaving

$$\sum_{r=1}^{n} \frac{1}{4r^2-1} = \frac{1}{2}\left(1 - \frac{1}{2n+1}\right)$$
$$= \frac{1}{2} \cdot \frac{2n}{2n+1}$$
$$= \frac{n}{2n+1}.$$

$\therefore \sum_{r=1}^{n} \frac{1}{4r^2-1} = \frac{n}{2n+1}$.

Part (ii) 2 marks

APPROACH

The given sum starts at $r = 11$ instead of $r = 1$. We get it from the result of (i) by taking $n \to \infty$ to get the sum to infinity, then subtracting off the first 10 terms (i.e. the sum up to $r = 10$), since $\sum_{r=11}^{\infty} u_r = \sum_{r=1}^{\infty} u_r - \sum_{r=1}^{10} u_r$, where $u_r = \frac{1}{4r^2-1}$.

As $n \to \infty$, $\frac{n}{2n+1} = \frac{1}{2+\frac{1}{n}} \to \frac{1}{2}$, so

$$\sum_{r=1}^{\infty} \frac{1}{4r^2-1} = \frac{1}{2}.$$

Using the result of (i) with $n = 10$,

$$\sum_{r=1}^{10} \frac{1}{4r^2-1} = \frac{10}{21}.$$

Hence

$$\sum_{r=11}^{\infty} \frac{1}{4r^2-1} = \frac{1}{2} - \frac{10}{21}$$
$$= \frac{21-20}{42}$$
$$= \frac{1}{42}.$$

$\therefore \sum_{r=11}^{\infty} \frac{1}{4r^2-1} = \frac{1}{42}$.

Question 7 8 marks

Differentiation · tangents · angle between lines

Part (i) 6 marks

Since $y = xe^{-x}$, we use the product rule to differentiate.

$$\begin{aligned}\frac{dy}{dx} &= (1)e^{-x} + x\left(-e^{-x}\right) \\ &= e^{-x}(1-x).\end{aligned}$$

At $x = 1$:

$$y = (1)e^{-1} = \frac{1}{e}, \qquad \frac{dy}{dx} = e^{-1}(1-1) = 0.$$

The tangent has gradient 0 and passes through $\left(1,\ \frac{1}{e}\right)$, so it is the horizontal line

$$y = \frac{1}{e}.$$

At $x = -1$:

$$y = (-1)e^{1} = -e, \qquad \frac{dy}{dx} = e^{1}\left(1-(-1)\right) = 2e.$$

Using $y - y_1 = m\left(x - x_1\right)$ with $(x_1, y_1) = (-1, -e)$ and $m = 2e$,

$$\begin{aligned}y - (-e) &= 2e\left(x-(-1)\right) \\ y + e &= 2e(x+1) \\ y &= 2ex + 2e - e \\ y &= 2ex + e.\end{aligned}$$

To check your answer: at $x = -1$, $2e(-1) + e = -e$, matching $y = -e$ found above. ✓

∴ the tangent at $x = 1$ is $y = \frac{1}{e}$, and the tangent at $x = -1$ is $y = 2ex + e$.

Part (ii) 2 marks

APPROACH

The acute angle between two lines of gradients m_1 and m_2 satisfies $\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$. Here one gradient is 0, so this simplifies a great deal.

Using the gradients found in (i), $m_1 = 0$ and $m_2 = 2e$,

$$\begin{aligned}\tan\theta &= \left|\frac{2e - 0}{1 + (0)(2e)}\right| \\ &= 2e.\end{aligned}$$

By GC, $\theta = \arctan(2e) = 79.6^\circ$ (1 d.p.), which is already acute.

∴ the acute angle between the tangents is 79.6° (1 d.p.).

Question 8 11 marks

Sequences and series · AP and GP

Part (a) 3 marks

Let S_{64} be the sum of the first 64 terms of the arithmetic series with first term a and common difference $2a$.

$$\begin{aligned} S_{64} &= \frac{64}{2}\{2a + (64-1)(2a)\} \\ &= 32(2a + 126a) \\ &= 32(128a) \\ &= 4096a. \end{aligned}$$

The kth term of the geometric series with first term a and common ratio 2 is $ar^{k-1} = a(2)^{k-1}$. Using the given equation,

$$a(2)^{k-1} = 4096a$$

Since $a \neq 0$, we divide both sides by a:

$$2^{k-1} = 4096$$

Since $4096 = 2^{12}$,

$$2^{k-1} = 2^{12}.$$

Comparing indices,

$$\begin{aligned} k - 1 &= 12 \\ k &= 13. \end{aligned}$$

$\therefore k = 13.$

Part (b) 4 marks

APPROACH

The sum of the first four terms is $f\left(1 + r + r^2 + r^3\right)$. Since $f \neq 0$, the bracket itself must be 0; we factorise it to find which real values of r are possible.

Since the sum of the first four terms is 0 and $f \neq 0$,

$$1 + r + r^2 + r^3 = 0$$

Grouping in pairs,

$$\begin{aligned} (1 + r) + r^2(1 + r) &= 0 \\ (1 + r)\left(1 + r^2\right) &= 0. \end{aligned}$$

Since $r \in \mathbb{R}$, $1 + r^2 > 0$ for every r, so $1 + r^2 \neq 0$. Hence

$$\begin{aligned} 1 + r &= 0, \\ r &= -1. \end{aligned}$$

Note: f itself is not constrained further by this equation — any nonzero real value of f gives a valid series once $r = -1$.

$\therefore r = -1$, and f may be any nonzero real number.

With $r = -1$, the sum of the first n terms is

$$\begin{aligned} S_n &= \frac{f\,(1 - r^n)}{1 - r} \\ &= \frac{f\,(1 - (-1)^n)}{1 - (-1)} \\ &= \frac{f\,(1 - (-1)^n)}{2}. \end{aligned}$$

If n is even, $(-1)^n = 1$, so $S_n = 0$. If n is odd, $(-1)^n = -1$, so

$$S_n = \frac{f\,(1 - (-1))}{2} = f.$$

$\therefore S_n = 0$ if n is even, and $S_n = f$ if n is odd.

To check your answer: with $r = -1$, the terms are $f,\ -f,\ f,\ -f,\ \ldots$, so partial sums alternate between f (odd n) and 0 (even n), matching the formula. ✓

Part (c) 4 marks

APPROACH

Let the first term be a (given negative) and the common difference d. The sum condition gives one linear equation in a and d. The product condition, $a(a+d)(a+2d)(a+3d) = 0$ with $a \neq 0$ (it is negative), forces one of the later three terms to be 0; we test each of the three cases against $a < 0$.

Let the arithmetic series have first term $a < 0$ and common difference d. The sum of the first four terms is 14:

$$\begin{aligned} \frac{4}{2}\{2a + 3d\} &= 14 \\ 2a + 3d &= 7. \quad (*) \end{aligned}$$

The product of the first four terms is 0:

$$a(a+d)(a+2d)(a+3d) = 0.$$

Since $a \neq 0$ (it is negative), exactly one of $a+d$, $a+2d$, $a+3d$ must be 0. We test each case against $(*)$ and the requirement $a < 0$.

Case $a + d = 0$, i.e. $d = -a$. Substituting into $(*)$:

$$\begin{aligned} 2a + 3(-a) &= 7 \\ -a &= 7 \\ a &= -7, \end{aligned}$$

which satisfies $a < 0$. Then $d = -a = 7$. Checking: the four terms are $-7,\ 0,\ 7,\ 14$, with product 0 . ✓ (valid)

Case $a + 2d = 0$, i.e. $d = -\frac{a}{2}$. Substituting into (*):

$$\begin{aligned} 2a + 3\left(-\frac{a}{2}\right) &= 7 \\ \frac{a}{2} &= 7 \\ a &= 14, \end{aligned}$$

which contradicts $a < 0$. (rejected)

Case $a + 3d = 0$, i.e. $d = -\frac{a}{3}$. Substituting into (*):

$$\begin{aligned} 2a + 3\left(-\frac{a}{3}\right) &= 7 \\ a &= 7, \end{aligned}$$

which again contradicts $a < 0$. (rejected)

Only the first case is valid, so $a = -7$ and $d = 7$. The 11th term is

$$\begin{aligned} a + 10d &= -7 + 10(7) \\ &= -7 + 70 \\ &= 63. \end{aligned}$$

$\therefore$ the 11th term is 63.

Question 9 11 marks

Complex numbers · polar form · roots of unity

Part (i)(a) 2 marks

Since $|w| = 1$, we have $w^{-1} = \overline{w}$. By De Moivre's theorem applied to $w = \cos\theta + i\sin\theta$,

$$\begin{aligned} \frac{1}{w} = w^{-1} &= \cos(-\theta) + i\sin(-\theta) \\ &= \cos\theta - i\sin\theta. \end{aligned}$$

Adding this to w,

$$\begin{aligned} w + \frac{1}{w} &= (\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta) \\ &= 2\cos\theta, \end{aligned}$$

which is real since θ is real. (shown)

Part (i)(b) 4 marks

APPROACH

$w - 1$ and $w + 1$ each split into a real and an imaginary part in θ. Rewriting each part with the half-angle identities $\cos\theta - 1 = -2\sin^2\frac{1}{2}\theta$, $\cos\theta + 1 = 2\cos^2\frac{1}{2}\theta$ and $\sin\theta = 2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$ turns both the numerator and the denominator into a common factor of $\left(\cos\frac{1}{2}\theta + i\sin\frac{1}{2}\theta\right)$, which then cancels.

$$
\begin{aligned}
w - 1 &= (\cos\theta - 1) + i\sin\theta \\
&= -2\sin^2\frac{1}{2}\theta + 2i\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta \\
&= 2\sin\frac{1}{2}\theta\left(-\sin\frac{1}{2}\theta + i\cos\frac{1}{2}\theta\right) \\
&= 2i\sin\frac{1}{2}\theta\left(\cos\frac{1}{2}\theta + i\sin\frac{1}{2}\theta\right),
\end{aligned}
$$

since $i\left(\cos\frac{1}{2}\theta + i\sin\frac{1}{2}\theta\right) = i\cos\frac{1}{2}\theta - \sin\frac{1}{2}\theta$.

Similarly,

$$
\begin{aligned}
w + 1 &= (\cos\theta + 1) + i\sin\theta \\
&= 2\cos^2\frac{1}{2}\theta + 2i\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta \\
&= 2\cos\frac{1}{2}\theta\left(\cos\frac{1}{2}\theta + i\sin\frac{1}{2}\theta\right).
\end{aligned}
$$

Dividing,

$$
\begin{aligned}
\frac{w-1}{w+1} &= \frac{2i\sin\frac{1}{2}\theta\left(\cos\frac{1}{2}\theta + i\sin\frac{1}{2}\theta\right)}{2\cos\frac{1}{2}\theta\left(\cos\frac{1}{2}\theta + i\sin\frac{1}{2}\theta\right)} \\
&= i\frac{\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta} \\
&= i\tan\frac{1}{2}\theta. \quad \text{(shown)}
\end{aligned}
$$

$\therefore k = i$.

To check your answer: at $\theta = \frac{\pi}{2}$, $w = i$, so $\frac{w-1}{w+1} = \frac{i-1}{i+1} = \frac{(i-1)(1-i)}{(i+1)(1-i)} = \frac{2i}{2} = i$, and $i\tan\frac{\pi}{4} = i(1) = i$. ✓

Part (ii) 5 marks

APPROACH

With $|z| = 1$ we have $z\overline{z} = 1$, so $\overline{z} = \frac{1}{z}$. This lets us compute $\left|\frac{z-3i}{1+3iz}\right|^2$ by multiplying numerator and denominator by their own conjugates and simplifying using $z\overline{z} = 1$, rather than substituting $z = \cos\phi + i\sin\phi$ and expanding trigonometric terms.

Since $|z| = 1$, $z\overline{z} = 1$. Consider the square of the modulus of the numerator:

$$
\begin{aligned}
|z - 3i|^2 &= (z - 3i)(\overline{z} + 3i) \\
&= z\overline{z} + 3iz - 3i\overline{z} - 9i^2 \\
&= 1 + 9 + 3i(z - \overline{z}) \\
&= 10 + 3i(z - \overline{z}).
\end{aligned}
$$

Now the square of the modulus of the denominator:

$$
\begin{aligned}
|1 + 3iz|^2 &= (1 + 3iz)(1 - 3i\overline{z}) \\
&= 1 - 3i\overline{z} + 3iz - 9i^2 z\overline{z} \\
&= 1 + 9(1) + 3i(z - \overline{z})
\end{aligned}
$$

$$= 10 + 3i\,(z - \overline{z})\,.$$

The two are equal, so

$$\left|\frac{z-3i}{1+3iz}\right|^2 = \frac{|z-3i|^2}{|1+3iz|^2} = 1.$$

$\therefore$ the modulus of $\frac{z-3i}{1+3iz}$ is 1.

Question 10 12 marks

Parametric equations · area · volume

Part (i) 2 marks

We first locate the cusp and the extreme points of C, so the sketch is accurate.

At $\theta = 0$: $\frac{dx}{d\theta} = 2a\,(\sin 2\theta - \sin\theta) = 0$ and $\frac{dy}{d\theta} = 2a\,(\cos\theta - \cos 2\theta) = 0$, so C has a cusp there, at $(a, 0)$.

At $\theta = \pi$: $x = a\,(2\cos\pi - \cos 2\pi) = a(-2-1) = -3a$, $y = 0$, giving the point $(-3a, 0)$.

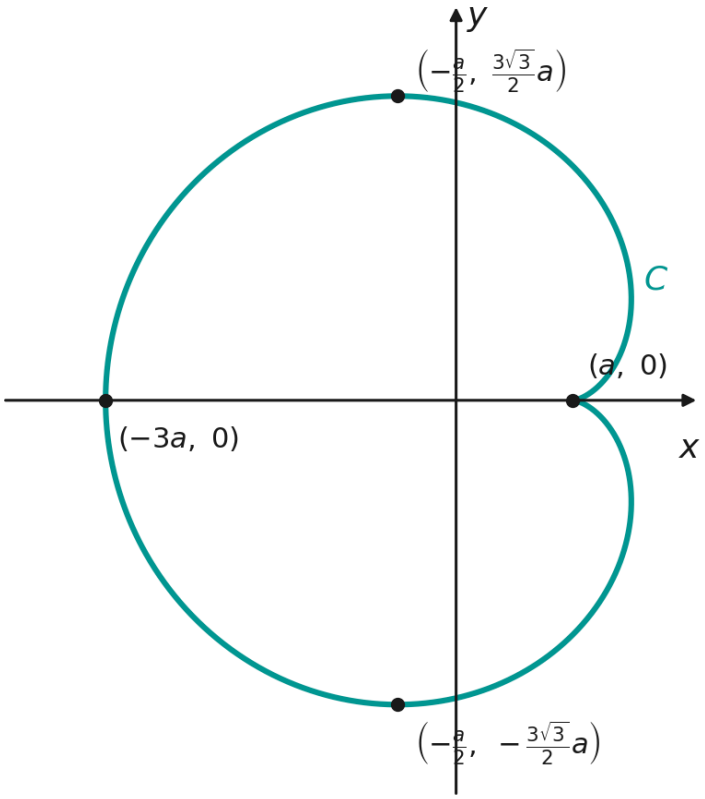
Setting $\frac{dy}{d\theta} = 0$ for the turning points off the axis,

$$\cos\theta = \cos 2\theta$$
$$= 2\cos^2\theta - 1,$$
$$2\cos^2\theta - \cos\theta - 1 = 0$$
$$(2\cos\theta + 1)\,(\cos\theta - 1) = 0.$$

So $\cos\theta = 1$ (the cusp, already found) or $\cos\theta = -\frac{1}{2}$, giving $\theta = \frac{2\pi}{3}$ or $\theta = \frac{4\pi}{3}$. At $\theta = \frac{2\pi}{3}$,

$$x = a\left(2\left(-\frac{1}{2}\right) - \left(-\frac{1}{2}\right)\right) = -\frac{1}{2}a, \qquad y = a\left(2\cdot\frac{\sqrt{3}}{2} - \left(-\frac{\sqrt{3}}{2}\right)\right) = \frac{3\sqrt{3}}{2}a,$$

giving the topmost point $\left(-\frac{a}{2},\ \frac{3\sqrt{3}}{2}a\right)$, with the mirror point $\left(-\frac{a}{2},\ -\frac{3\sqrt{3}}{2}a\right)$ at $\theta = \frac{4\pi}{3}$ below the x-axis.



C is a cardioid, symmetric about the x-axis (shown algebraically in (ii)). Its line of symmetry has Cartesian equation

$$y = 0.$$

Part (ii) 2 marks

APPROACH

C meets the x-axis where $y = 0$; factorising the resulting equation in θ gives all the parameter values at once.

C meets the x-axis where $y = 0$:

$$a\left(2\sin\theta - \sin 2\theta\right) = 0$$
$$2\sin\theta - 2\sin\theta\cos\theta = 0$$
$$2\sin\theta\left(1 - \cos\theta\right) = 0.$$

So $\sin\theta = 0$ or $\cos\theta = 1$. For $0 \le \theta \le 2\pi$, $\sin\theta = 0$ gives $\theta = 0,\ \pi,\ 2\pi$, and $\cos\theta = 1$ gives $\theta = 0,\ 2\pi$ (already included).

$\therefore$ C meets the x-axis at $\theta = 0,\ \pi,\ 2\pi$.

Note: $\theta = 0$ and $\theta = 2\pi$ both give the same point $(a, 0)$, the cusp where the curve closes up; $\theta = \pi$ gives the distinct point $(-3a, 0)$.

Part (iii) 3 marks

> **APPROACH**
>
> For $0 \le \theta \le \pi$, $y \ge 0$ traces the upper half of C from the cusp $(a, 0)$ back to $(-3a, 0)$, with x decreasing throughout (since the curve is traversed leftwards along the top). The area under this arc is $-\int y \, \frac{dx}{d\theta} \, d\theta$, the minus sign compensating for x decreasing as θ increases.

By symmetry about the x-axis, the upper half of C (where $y \ge 0$) is traced for $0 \le \theta \le \pi$, from $\theta_1 = 0$ to $\theta_2 = \pi$. Differentiating,

$$\frac{dx}{d\theta} = a\left(-2\sin\theta + 2\sin 2\theta\right) = 2a\left(\sin 2\theta - \sin\theta\right).$$

Since x decreases as θ runs from 0 to π, the area enclosed by the x-axis and this arc is

$$\text{Area} = -\int_0^{\pi} y \, \frac{dx}{d\theta} \, d\theta.$$

Now

$$\begin{aligned} y\,\frac{dx}{d\theta} &= a\left(2\sin\theta - \sin 2\theta\right) \cdot 2a\left(\sin 2\theta - \sin\theta\right) \\ &= 2a^2\left(2\sin\theta - \sin 2\theta\right)\left(\sin 2\theta - \sin\theta\right) \\ &= 2a^2\left(3\sin\theta\sin 2\theta - 2\sin^2\theta - \sin^2 2\theta\right) \\ &= a^2\left(6\sin\theta\sin 2\theta - 4\sin^2\theta - 2\sin^2 2\theta\right). \end{aligned}$$

So

$$\begin{aligned} \text{Area} &= -\int_0^{\pi} a^2\left(6\sin\theta\sin 2\theta - 4\sin^2\theta - 2\sin^2 2\theta\right) \, d\theta \\ &= \int_{\theta_1}^{\theta_2} a^2\left(4\sin^2\theta - 6\sin\theta\sin 2\theta + 2\sin^2 2\theta\right) \, d\theta, \end{aligned}$$

with $\theta_1 = 0$ and $\theta_2 = \pi$. (shown)

Part (iv) 5 marks

> **APPROACH**
>
> Each term of the integrand reduces with a standard identity: $\sin^2\theta$ and $\sin^2 2\theta$ via the double-angle cosine formula, and $\sin\theta\sin 2\theta$ via the product-to-sum formula. The total area is then twice the area of the upper half, by the symmetry found in (i).

Using $\sin^2\theta = \frac{1}{2}\left(1 - \cos 2\theta\right)$ and $\sin^2 2\theta = \frac{1}{2}\left(1 - \cos 4\theta\right)$,

$$\int_0^{\pi} 4\sin^2\theta \, d\theta = \int_0^{\pi} 2\left(1 - \cos 2\theta\right) \, d\theta = 2\pi,$$

$$\int_0^{\pi} 2\sin^2 2\theta \, d\theta = \int_0^{\pi} \left(1 - \cos 4\theta\right) \, d\theta = \pi.$$

Using $\sin\theta\sin 2\theta = \frac{1}{2}\left(\cos\theta - \cos 3\theta\right)$,

$$\int_0^{\pi} 6\sin\theta\sin 2\theta \, d\theta = \int_0^{\pi} 3\left(\cos\theta - \cos 3\theta\right) \, d\theta$$

$$= 3\left[\sin\theta - \frac{1}{3}\sin 3\theta\right]_0^{\pi}$$
$$= 0.$$

So the area of the upper half is

$$\int_0^{\pi} a^2\left(4\sin^2\theta - 6\sin\theta\sin 2\theta + 2\sin^2 2\theta\right)\,d\theta = a^2(2\pi - 0 + \pi) = 3\pi a^2.$$

By the symmetry of C about the x-axis, the lower half encloses the same area, so the total area enclosed by C is

$$2 \times 3\pi a^2 = 6\pi a^2.$$

To check your answer: a numerical shoelace computation on $1000+$ plotted points of C with $a = 1$ gives area ≈ 18.8496, matching $6\pi \approx 18.8496$. ✓

$\therefore$ the total area enclosed by C is $6\pi a^2$.

Question 11 13 marks

Differential equations · Newton's law of cooling

Part (i)(a) 6 marks

Let θ be the temperature of the water, in $^\circ C$, t minutes after it is placed in the room. Since the rate of decrease of θ is proportional to the difference between θ and the room temperature $16\,^\circ C$,

$$\frac{d\theta}{dt} = -k(\theta - 16),$$

where k is a positive constant.

Since the water is cooling towards $16\,^\circ C$ from above, $\theta > 16$ throughout, so $\theta - 16 > 0$. Separating variables,

$$\int \frac{1}{\theta - 16}\,d\theta = \int -k\,dt$$
$$\ln(\theta - 16) = -kt + C.$$

So

$$\theta - 16 = e^{-kt+C}$$
$$= Ae^{-kt},$$

where $A = e^C$.

Using the initial condition $\theta = 80$ when $t = 0$,

$$80 - 16 = Ae^0$$
$$A = 64.$$

So $\theta = 16 + 64e^{-kt}$. Using the given condition $\theta = 32$ when $t = 30$,

$$32 - 16 = 64e^{-30k}$$

$$e^{-30k} = \frac{1}{4}$$
$$-30k = \ln \frac{1}{4}$$
$$k = \frac{\ln 4}{30} = \frac{\ln 2}{15}.$$

$\therefore$

$$\theta = 16 + 64e^{-\frac{\ln 2}{15}t}$$

as an exact function of t.

Part (i)(b) 1 mark

$$\theta(45) = 16 + 64e^{-\frac{\ln 2}{15}(45)}$$
$$= 16 + 64e^{-3\ln 2}$$
$$= 16 + 64\left(2^{-3}\right)$$
$$= 16 + 8$$
$$= 24.$$

$\therefore$ the temperature of the water after 45 minutes is $24\,^{\circ}C$.

Part (ii) 6 marks

APPROACH

"Rate of change of thickness inversely proportional to thickness" gives $\frac{dT}{dt} = \frac{c}{T}$. This is separable, and $T = 0$ at $t = 0$ pins down the constant of integration exactly (no logarithm is involved, since the variable being integrated is T itself, not $T - \text{constant}$).

Let T cm be the thickness of the ice t minutes after freezing commences. Since the rate of change of T is inversely proportional to T,

$$\frac{dT}{dt} = \frac{c}{T},$$

where c is a positive constant. Separating variables,

$$\int T\,dT = \int c\,dt$$
$$\frac{T^2}{2} = ct + D.$$

Using $T = 0$ when $t = 0$,

$$0 = c(0) + D$$
$$D = 0.$$

So $T^2 = 2ct$. Using the given condition $T = 1$ when $t = 60$,

$$1 = 2c(60)$$
$$c = \frac{1}{120}.$$

So

$$T^2 = \frac{t}{60},$$

and since $T \geq 0$,

$$T = \sqrt{\frac{t}{60}}.$$

The ice is safe to skate on once $T = 3$:

$$\begin{aligned} 3 &= \sqrt{\frac{t}{60}} \\ 9 &= \frac{t}{60} \\ t &= 540. \end{aligned}$$

To check your answer: $T(60)^2 = 60/60 = 1$, matching the given data point, and $T(540) = \sqrt{540/60} = \sqrt{9} = 3$. ✓

$\therefore$ the ice is first safe to skate on 540 minutes (i.e. 9 hours) after freezing commences.

thelearningcave.com.sg

Question 12 13 marks

Vectors · lines · planes · reflection

Part (i) 5 marks

APPROACH

Q is the point where the line through P in the direction of the ray meets the top plane; R is the point where the line through S, parallel to that same direction (since PQ is parallel to RS), meets the base plane. Both are found by substituting a general point of the line into the plane equation and solving for the parameter.

The line along PQ passes through $P(2, 2, 4)$ in direction $\begin{pmatrix} -2 \\ -3 \\ -6 \end{pmatrix}$, so a general point on it is

$$\begin{pmatrix} 2 - 2\lambda \\ 2 - 3\lambda \\ 4 - 6\lambda \end{pmatrix}.$$

Since Q lies on the plane $x + y + z = 1$,

$$\begin{aligned} (2 - 2\lambda) + (2 - 3\lambda) + (4 - 6\lambda) &= 1 \\ 8 - 11\lambda &= 1 \\ \lambda &= \frac{7}{11}. \end{aligned}$$

So

$$Q = \left(2 - \frac{14}{11},\ 2 - \frac{21}{11},\ 4 - \frac{42}{11}\right) = \left(\frac{8}{11},\ \frac{1}{11},\ \frac{2}{11}\right).$$

Since PQ is parallel to RS, the line along RS also has direction $\begin{pmatrix}-2\\-3\\-6\end{pmatrix}$, so a general point of the line through $S(-5,-6,-7)$ is

$$\begin{pmatrix}-5-2\mu\\-6-3\mu\\-7-6\mu\end{pmatrix}.$$

Since R lies on the plane $x+y+z=-9$,

$$(-5-2\mu)+(-6-3\mu)+(-7-6\mu) = -9$$
$$-18-11\mu = -9$$
$$\mu = -\frac{9}{11}.$$

So

$$R = \left(-5+\frac{18}{11},\ -6+\frac{27}{11},\ -7+\frac{54}{11}\right) = \left(-\frac{37}{11},\ -\frac{39}{11},\ -\frac{23}{11}\right).$$

To check your answer: the coordinates of R sum to $-\frac{37}{11}-\frac{39}{11}-\frac{23}{11} = -9$, matching the base plane equation. ✓

$\therefore Q = \left(\frac{8}{11},\ \frac{1}{11},\ \frac{2}{11}\right)$ and $R = \left(-\frac{37}{11},\ -\frac{39}{11},\ -\frac{23}{11}\right)$.

Part (ii) 3 marks

Let $\mathbf{n} = \begin{pmatrix}1\\1\\1\end{pmatrix}$ be a normal to both the top and (parallel) base planes. Since θ and β are the acute angles between the rays PQ, QR and this normal, we use the modulus of the dot product.

$$\overrightarrow{PQ} = Q - P = \begin{pmatrix}-14/11\\-21/11\\-42/11\end{pmatrix}, \qquad \overrightarrow{QR} = R - Q = \begin{pmatrix}-45/11\\-40/11\\-25/11\end{pmatrix}.$$

$$\left|\overrightarrow{PQ}\right| = \frac{1}{11}\sqrt{14^2+21^2+42^2} = \frac{1}{11}\sqrt{2401} = \frac{49}{11}, \qquad \overrightarrow{PQ}\cdot\mathbf{n} = -\frac{14+21+42}{11} = -7.$$

Since θ is acute,

$$\cos\theta = \frac{\left|\overrightarrow{PQ}\cdot\mathbf{n}\right|}{\left|\overrightarrow{PQ}\right|\,|\mathbf{n}|}$$
$$= \frac{7}{(49/11)\sqrt{3}}$$
$$= \frac{77}{49\sqrt{3}}$$
$$= \frac{11}{7\sqrt{3}}$$

$$= \frac{11\sqrt{3}}{21}.$$

Similarly,

$$\left|\overrightarrow{QR}\right| = \frac{1}{11}\sqrt{45^2 + 40^2 + 25^2} = \frac{1}{11}\sqrt{4250} = \frac{5\sqrt{170}}{11}, \qquad \overrightarrow{QR} \cdot \mathbf{n} = -\frac{45 + 40 + 25}{11} = -10.$$

$$\begin{aligned}
\cos\beta &= \frac{\left|\overrightarrow{QR} \cdot \mathbf{n}\right|}{\left|\overrightarrow{QR}\right| |\mathbf{n}|} \\
&= \frac{10}{\left(5\sqrt{170}/11\right)\sqrt{3}} \\
&= \frac{10 \times 11}{5\sqrt{170}\sqrt{3}} \\
&= \frac{22}{\sqrt{510}} \\
&= \frac{11\sqrt{510}}{255}.
\end{aligned}$$

By GC, $\cos\theta = 0.907$ and $\cos\beta = 0.974$ (3 s.f.).

$\therefore \cos\theta = \frac{11\sqrt{3}}{21} = 0.907$ and $\cos\beta = \frac{11\sqrt{510}}{255} = 0.974$ (3 s.f.).

Part (iii) 3 marks

APPROACH

The prism's two faces are parallel planes; the thickness measured along the normal is simply the perpendicular distance between them, found from the plane constants divided by $|\mathbf{n}|$.

The top and base planes, $x + y + z = 1$ and $x + y + z = -9$, are both normal to $\mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, so the perpendicular distance between them is

$$\begin{aligned}
\text{Thickness} &= \frac{|1 - (-9)|}{|\mathbf{n}|} \\
&= \frac{10}{\sqrt{3}} \\
&= \frac{10\sqrt{3}}{3}.
\end{aligned}$$

$\therefore$ the thickness of the prism measured in the direction of the normal at Q is $\frac{10\sqrt{3}}{3} = 5.77$ (3 s.f.).

Part (iv) 1 mark

Using $\sin\theta = k\sin\beta$ with θ, β acute (so $\sin\theta, \sin\beta > 0$),

$$\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \frac{121(3)}{441}} = \sqrt{\frac{78}{441}} = \frac{\sqrt{78}}{21},$$

$$\sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{1-\frac{121(510)}{255^2}} = \sqrt{\frac{13}{255}}.$$

So

$$\begin{aligned} k &= \frac{\sin\theta}{\sin\beta} \\ &= \frac{\sqrt{78}/21}{\sqrt{13/255}} \\ &= \frac{\sqrt{78}}{21}\sqrt{\frac{255}{13}} \\ &= \frac{\sqrt{78\times 255/13}}{21} \\ &= \frac{\sqrt{1530}}{21} \\ &= \frac{\sqrt{170}}{7}, \end{aligned}$$

using $78/13 = 6$ so $78\times 255/13 = 6\times 255 = 1530 = 9\times 170$, and $\sqrt{1530} = 3\sqrt{170}$.

By GC, $k = 1.86$ (3 s.f.).

$\therefore k = \frac{\sqrt{170}}{7} = 1.86$ (3 s.f.).

Part (v) 1 mark

Since $\sin\theta = k\sin\beta$ and $\theta, \beta \in \left(0, \frac{\pi}{2}\right)$, where $\sin$ is increasing, $\beta > \theta$ means $\sin\beta > \sin\theta$. Then

$$k = \frac{\sin\theta}{\sin\beta} < 1.$$

$\therefore$ if $\beta > \theta$, the refractive index $k < 1$.

thelearningcave.com.sg
Physics. Math. Chemistry.