TEN-YEAR SERIES · WORKED SOLUTIONS

2018 A-Level H2 Mathematics Paper 2

10 questions · 100 marks

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Physics. Math. Chemistry.

Question 1 6 marks

Differential equations · separable equations

Part (i) 4 marks

Since $\frac{dy}{dx} = \left(\frac{1}{3}y - 15\right)^{\frac{1}{3}}$ is a function of y only, we separate variables.

$$\int \frac{dy}{\left(\frac{1}{3}y - 15\right)^{\frac{1}{3}}} = \int dx$$

Let $u = \frac{1}{3}y - 15$, so that $y = 3u + 45$ and $dy = 3\,du$.

$$3\int u^{-\frac{1}{3}}\,du = \int dx$$

Integrating both sides,

$$\frac{9}{2}u^{\frac{2}{3}} = x + C$$

Substituting the initial condition $(0, 69)$: when $x = 0$, $y = 69$, so $u = \frac{1}{3}(69) - 15 = 8$.

$$\begin{aligned}\frac{9}{2}(8)^{\frac{2}{3}} &= C \\ &= \frac{9}{2}(4) \\ &= 18\end{aligned}$$

So $C = 18$.

So $\frac{9}{2}u^{\frac{2}{3}} = x + 18$, giving $u^{\frac{2}{3}} = \frac{2}{9}(x + 18)$, so $u = \left[\frac{2}{9}(x + 18)\right]^{\frac{3}{2}}$.

Since $y = 3u + 45$,

$$f(x) = 45 + \frac{2\sqrt{2}}{9}(x + 18)^{\frac{3}{2}}$$

To check your answer: $f(0) = 45 + \frac{2\sqrt{2}}{9}(18)^{\frac{3}{2}} = 45 + 24 = 69$, which matches the given point. ✓

$\therefore f(x) = 45 + \frac{2\sqrt{2}}{9}(x + 18)^{\frac{3}{2}}$.

Part (ii) 2 marks

APPROACH

The gradient is given directly in terms of y, so it is quicker to find y first from the gradient condition, then use the relation $\frac{9}{2}u^{\frac{2}{3}} = x + 18$ from part (i) to recover x, rather than differentiating $f(x)$.

Using the given gradient,

$$\left(\frac{1}{3}y - 15\right)^{\frac{1}{3}} = 4$$

$$\Rightarrow \frac{1}{3}y - 15 = 64$$

$$\Rightarrow y = 237$$

So $u = 64$ at this point, and $u^{\frac{2}{3}} = 16$. Substituting into $\frac{9}{2}u^{\frac{2}{3}} = x + 18$,

$$\frac{9}{2}(16) = x + 18$$
$$\Rightarrow 72 = x + 18$$
$$\Rightarrow x = 54$$

$\therefore$ the required point is $(54,\ 237)$.

Question 2 12 marks

Complex numbers · roots of polynomials · loci

Part (a) 5 marks

Since s and t are real, the coefficients of the quartic are real, so complex roots occur in conjugate pairs. As $2 - 3i$ is a root, $2 + 3i$ is also a root.

$$(x - (2 - 3i))\,(x - (2 + 3i))$$
$$= ((x - 2) + 3i)\,((x - 2) - 3i)$$
$$= (x - 2)^2 + 9$$
$$= x^2 - 4x + 13$$

APPROACH

The quartic has leading coefficient 4, and $x^2 - 4x + 13$ (the factor from the known conjugate pair) is monic, so the remaining quadratic factor must also be monic. We write the quartic as $4\left(x^2 - 4x + 13\right)\left(x^2 + bx + c\right)$ and compare coefficients to find b and c, then s and t.

$$4x^4 - 20x^3 + sx^2 - 56x + t = 4\left(x^2 - 4x + 13\right)\left(x^2 + bx + c\right)$$

Expanding the right side,

$$4\left(x^2 - 4x + 13\right)\left(x^2 + bx + c\right)$$
$$= 4x^4 + 4bx^3 + 4cx^2 - 16x^3 - 16bx^2 - 16cx + 52x^2 + 52bx + 52c$$
$$= 4x^4 + (4b - 16)x^3 + (4c - 16b + 52)x^2 + (52b - 16c)x + 52c$$

Comparing the coefficient of x^3 with -20,

$$4b - 16 = -20$$
$$\Rightarrow b = -1$$

Comparing the coefficient of x with -56,

$$52b - 16c = -56$$
$$\Rightarrow 52(-1) - 16c = -56$$
$$\Rightarrow c = \frac{1}{4}$$

So the other quadratic factor is $x^2 - x + \frac{1}{4} = \left(x - \frac{1}{2}\right)^2$, giving a repeated root $x = \frac{1}{2}$.

Comparing the remaining coefficients,

$$s = 4c - 16b + 52 = 4\left(\frac{1}{4}\right) - 16(-1) + 52 = 69$$

$$t = 52c = 52\left(\frac{1}{4}\right) = 13$$

To check your answer: with $s = 69$, $t = 13$, substituting $x = \frac{1}{2}$ into $4x^4 - 20x^3 + 69x^2 - 56x + 13$ gives $\frac{1}{4} - \frac{5}{2} + \frac{69}{4} - 28 + 13 = 0$. ✓

$\therefore$ the other roots are $x = 2 + 3i$ and $x = \frac{1}{2}$ (repeated), with $s = 69$ and $t = 13$.

Part (b)(i) 3 marks

APPROACH

A non-calculator method means we factorise $w^3 - 27$ using the known root $w = 3$, rather than solving $w^3 = 27$ directly.

Since $w = 3$ is a root of $w^3 - 27 = 0$, $(w - 3)$ is a factor.

$$w^3 - 27 = (w - 3)\left(w^2 + 3w + 9\right)$$

The other values of w satisfy $w^2 + 3w + 9 = 0$. By the quadratic formula,

$$w = \frac{-3 \pm \sqrt{9 - 36}}{2}$$

$$= \frac{-3 \pm \sqrt{-27}}{2}$$

$$= \frac{-3 \pm 3i\sqrt{3}}{2}$$

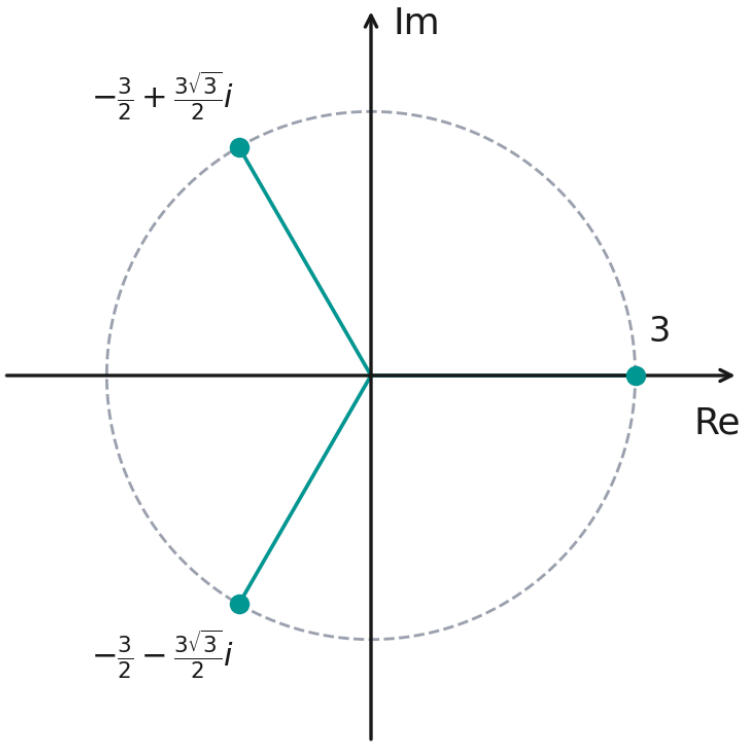
$\therefore$ the other values are $w = -\frac{3}{2} + \frac{3\sqrt{3}}{2}i$ and $w = -\frac{3}{2} - \frac{3\sqrt{3}}{2}i$.

Part (b)(ii) 2 marks

For each value of w, $|w| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{3\sqrt{3}}{2}\right)^2} = \sqrt{\frac{9}{4} + \frac{27}{4}} = 3$, consistent with all cube roots of 27 having modulus $27^{1/3} = 3$.

Since $\tan(\arg w) = \frac{3\sqrt{3}/2}{-3/2} = -\sqrt{3}$ and the point lies in the second quadrant (for the $+$ case) or third quadrant (for the - case), the arguments are $\frac{2\pi}{3}$ and $-\frac{2\pi}{3}$.

$$w = 3, \quad w = 3\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right), \quad w = 3\left(\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right)$$



Note: the three points lie on a circle of radius 3 centred at the origin, spaced $\frac{2\pi}{3}$ apart, since they are the three cube roots of 27.

Part (b)(iii) 2 marks

APPROACH

Rather than adding and multiplying the surd forms directly, we use the fact that the three values of w are the roots of $w^3 - 27 = 0$, so their sum and product follow from the coefficients.

Since $w^3 + 0w^2 + 0w - 27 = 0$ has roots 3, $-\frac{3}{2}+\frac{3\sqrt{3}}{2}i$, $-\frac{3}{2}-\frac{3\sqrt{3}}{2}i$,

$$\text{Sum of roots} = -\frac{0}{1} = 0$$

$$\text{Product of roots} = -\frac{-27}{1} = 27$$

To check your answer: $3\left(-\frac{3}{2}+\frac{3\sqrt{3}}{2}i\right)\left(-\frac{3}{2}-\frac{3\sqrt{3}}{2}i\right) = 3\left(\frac{9}{4}+\frac{27}{4}\right) = 3(9) = 27.$ ✓

$\therefore$ the sum of all possible values of w is 0 and the product is 27.

Question 3 12 marks

Vectors · planes · pyramid geometry

Part (i) 1 mark

Since ABCD is a parallelogram, the diagonals AC and BD bisect each other, so the midpoint of AC equals the midpoint of BD.

$$\frac{1}{2}\left(\overrightarrow{OA}+\overrightarrow{OC}\right)=\frac{1}{2}\left(\overrightarrow{OB}+\overrightarrow{OD}\right)$$
$$\Rightarrow \overrightarrow{OD}=\overrightarrow{OA}+\overrightarrow{OC}-\overrightarrow{OB}$$
$$=\begin{pmatrix}5\\-4\\1\end{pmatrix}+\begin{pmatrix}-5\\4\\2\end{pmatrix}-\begin{pmatrix}5\\4\\0\end{pmatrix}$$
$$=\begin{pmatrix}-5\\-4\\3\end{pmatrix}$$

$\therefore D=(-5,\ -4,\ 3)$.

Part (ii) 3 marks

APPROACH

The plane BCE is determined by the two points B, C and the apex E. We find two direction vectors in the plane, take their cross product for a normal, then use one point to fix the constant.

$$\overrightarrow{BC}=\begin{pmatrix}-5\\4\\2\end{pmatrix}-\begin{pmatrix}5\\4\\0\end{pmatrix}=\begin{pmatrix}-10\\0\\2\end{pmatrix},\quad \overrightarrow{BE}=\begin{pmatrix}0\\0\\10\end{pmatrix}-\begin{pmatrix}5\\4\\0\end{pmatrix}=\begin{pmatrix}-5\\-4\\10\end{pmatrix}$$

A normal to the plane is

$$\overrightarrow{n}=\overrightarrow{BC}\times\overrightarrow{BE}$$
$$=\begin{pmatrix}-10\\0\\2\end{pmatrix}\times\begin{pmatrix}-5\\-4\\10\end{pmatrix}$$
$$=\begin{pmatrix}(0)(10)-(2)(-4)\\(2)(-5)-(-10)(10)\\(-10)(-4)-(0)(-5)\end{pmatrix}$$
$$=\begin{pmatrix}8\\90\\40\end{pmatrix}$$

Dividing by the common factor 2, we may take $\overrightarrow{n}=\begin{pmatrix}4\\45\\20\end{pmatrix}$. Using point B,

$$\overrightarrow{n}\cdot\overrightarrow{OB}=4(5)+45(4)+20(0)=200$$

$\therefore$ the cartesian equation of face BCE is $4x + 45y + 20z = 200$.

To check your answer: substituting $C(-5, 4, 2)$: $4(-5) + 45(4) + 20(2) = -20 + 180 + 40 = 200$. ✓

Part (iii) 3 marks

APPROACH

The angle between the two planes equals the angle between their normal vectors (taking the acute angle). We already have the normal to face BCE; we now find a normal to the base ABCD using $\overrightarrow{AB}$ and $\overrightarrow{AD}$, then apply the formula for the angle between two planes.

$$\overrightarrow{AB} = \begin{pmatrix} 0 \\ 8 \\ -1 \end{pmatrix}, \quad \overrightarrow{AD} = \begin{pmatrix} -10 \\ 0 \\ 2 \end{pmatrix}$$

A normal to the base is

$$\overrightarrow{m} = \overrightarrow{AB} \times \overrightarrow{AD} = \begin{pmatrix} (8)(2) - (-1)(0) \\ (-1)(-10) - (0)(2) \\ (0)(0) - (8)(-10) \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \\ 40 \end{pmatrix}$$

Let θ be the angle between the two planes. Using the normals $\overrightarrow{n} = \begin{pmatrix} 4 \\ 45 \\ 20 \end{pmatrix}$ and $\overrightarrow{m} = \begin{pmatrix} 8 \\ 5 \\ 40 \end{pmatrix}$,

$$\cos\theta = \frac{\left|\overrightarrow{n} \cdot \overrightarrow{m}\right|}{\left|\overrightarrow{n}\right| \left|\overrightarrow{m}\right|}$$

$$= \frac{|4(8) + 45(5) + 20(40)|}{\sqrt{4^2 + 45^2 + 20^2}\sqrt{8^2 + 5^2 + 40^2}}$$

$$= \frac{1057}{\sqrt{2441}\sqrt{1689}}$$

By GC, $\theta = 58.6°$ (1 d.p.).

$\therefore$ the angle between face BCE and the base is $58.6°$.

Part (iv) 5 marks

APPROACH

The midpoint of AD is a point not on face BCE; its shortest distance to that plane is the perpendicular distance, found directly from the cartesian equation of the plane using the normal vector from part (ii).

Let M be the midpoint of AD.

$$M = \frac{1}{2}\left(\begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix} + \begin{pmatrix} -5 \\ -4 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ -4 \\ 2 \end{pmatrix}$$

Using the plane $4x + 45y + 20z = 200$ with normal $\overrightarrow{n} = \begin{pmatrix} 4 \\ 45 \\ 20 \end{pmatrix}$, $\left|\overrightarrow{n}\right| = \sqrt{2441}$,

$$\begin{aligned} \text{Distance} &= \frac{|4(0) + 45(-4) + 20(2) - 200|}{\sqrt{2441}} \\ &= \frac{|-180 + 40 - 200|}{\sqrt{2441}} \\ &= \frac{340}{\sqrt{2441}} \end{aligned}$$

By GC, this is 6.88 (3 s.f.).

$\therefore$ the shortest distance from the midpoint of AD to face BCE is $\frac{340}{\sqrt{2441}} = 6.88$ units.

Question 4 10 marks

Maclaurin series · standard series

Part (i) 9 marks

APPROACH

Rather than differentiating $\ln(\cos 2x)$ repeatedly, it is quicker to substitute the known Maclaurin series for $\cos x$ (from MF26) into $\cos 2x$, then substitute that into the known series for $\ln(1+v)$, keeping terms only up to x^6.

From MF26, $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$, so

$$\begin{aligned} &\cos 2x \\ &= 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \ldots \\ &= 1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \ldots \end{aligned}$$

Let $v = -2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6$, so that $\cos 2x = 1 + v$. From MF26, $\ln(1+v) = v - \frac{v^2}{2} + \frac{v^3}{3} - \ldots$.
Keeping terms up to x^6,

$$v^2 = \left(-2x^2 + \frac{2}{3}x^4\right)^2 + O\left(x^8\right) = 4x^4 - \frac{8}{3}x^6 + O\left(x^8\right)$$
$$v^3 = \left(-2x^2\right)^3 + O\left(x^8\right) = -8x^6 + O\left(x^8\right)$$

So

$$\begin{aligned} &\ln(\cos 2x) \\ &= v - \frac{v^2}{2} + \frac{v^3}{3} \\ &= \left(-2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6\right) - \frac{1}{2}\left(4x^4 - \frac{8}{3}x^6\right) + \frac{1}{3}\left(-8x^6\right) \\ &= -2x^2 + \left(\frac{2}{3} - 2\right)x^4 + \left(-\frac{4}{45} + \frac{4}{3} - \frac{8}{3}\right)x^6 \end{aligned}$$

$$= -2x^2 - \frac{4}{3}x^4 - \frac{64}{45}x^6$$

$\therefore \ln(\cos 2x) \approx -2x^2 - \frac{4}{3}x^4 - \frac{64}{45}x^6$.

Since $\ln(\cos 2x)$ is only defined where $\cos 2x > 0$, and on $0 \le x \le \frac{1}{4}\pi$ we have $2x \in \left[0, \frac{\pi}{2}\right]$ with $\cos 2x = 0$ only at $x = \frac{1}{4}\pi$,

$\therefore$ the expansion is not valid at $x = \frac{\pi}{4}$.

Part (ii)

Dividing the expansion by x^2,

$$\frac{\ln(\cos 2x)}{x^2} \approx -2 - \frac{4}{3}x^2 - \frac{64}{45}x^4$$

Integrating term by term,

$$\int \frac{\ln(\cos 2x)}{x^2}\,dx$$
$$\approx \int \left(-2 - \frac{4}{3}x^2 - \frac{64}{45}x^4\right) dx$$
$$= -2x - \frac{4}{9}x^3 - \frac{64}{225}x^5 + C$$

Using this approximate antiderivative between $x = 0$ and $x = 0.5$ (the series value at $x = 0$ is 0, matching the removable behaviour of $\frac{\ln(\cos 2x)}{x^2}$ there),

$$\int_0^{0.5} \frac{\ln(\cos 2x)}{x^2}\,dx$$
$$\approx \left[-2x - \frac{4}{9}x^3 - \frac{64}{225}x^5\right]_0^{0.5}$$
$$= -1 - \frac{1}{18} - \frac{2}{225}$$
$$= -\frac{479}{450}$$

$\therefore \int_0^{0.5} \frac{\ln(\cos 2x)}{x^2}\,dx \approx -\frac{479}{450} = -1.0644$ (4 d.p.).

Part (iii) 1 mark

By GC, $\int_0^{0.5} \frac{\ln(\cos 2x)}{x^2}\,dx = -1.0670$ (4 d.p.).

Note: this GC value is obtained directly from the original function (the GC evaluates the improper-looking integral fine, since $\frac{\ln(\cos 2x)}{x^2} \to -2$ as $x \to 0$), and is close to but not identical to the series-based approximation in (ii), since (ii) drops terms in x^8 and higher.

Question 5 7 marks

Hypothesis testing · test of a mean

Part (i) 2 marks

Since the distribution of the time to failure of an individual fan is not stated to be normal, we cannot assume that $\overline{X}$ is normally distributed for a small sample. Taking a sample of at least 30 fans lets the manager invoke the Central Limit Theorem, so that the sample mean $\overline{X}$ is approximately normally distributed whatever the distribution of an individual fan's time to failure, which is what the hypothesis test requires.

The fans should be chosen randomly (e.g. by simple random sampling) from the manufacturer's production, so that the sample is representative of all such fans and the test is not biased by, say, testing only fans from one batch or one production run.

Part (ii) 2 marks

Let μ be the population mean time to failure, in hours, of this type of fan (under the accelerated testing procedure). The quality control manager suspects the MTTF is less than the claimed value, so the test is

$$H_0 : \mu = 65000$$
$$H_1 : \mu < 65000$$

carried out at the 5% significance level.

Part (iii) 3 marks

APPROACH

We are not given the variance used in the test — instead we are told the *conclusion* of the test (do not reject H_0), and asked to work backwards for the range of variance values consistent with that conclusion. The population variance is unknown, so let σ^2 denote the (unbiased) estimate of it used in the test statistic. Since $n = 43$ is large, by the Central Limit Theorem, under H_0,

$$Z = \frac{\overline{X} - 65000}{\sigma/\sqrt{43}} \sim N(0,\ 1) \text{ approximately.}$$

This is a lower-tailed test at the 5% level, so H_0 is rejected if the test statistic $z \leq -1.6449$ (by GC, the critical value is $-invNorm(0.95) = -1.6449$, i.e. $z_{0.05} = 1.6449$).

Since the manager finds no reason to reject H_0, the test statistic must satisfy $z > -1.6449$. Using $\overline{x} = 64230$,

$$z = \frac{64230 - 65000}{\sigma/\sqrt{43}} = \frac{-770\sqrt{43}}{\sigma}.$$

Substituting into the non-rejection condition:

$$\frac{-770\sqrt{43}}{\sigma} > -1.6449$$

$$\Rightarrow \frac{770\sqrt{43}}{\sigma} < 1.6449 \text{ (multiplying by } -1\text{, so the inequality reverses)}$$

$$\Rightarrow \sigma > \frac{770\sqrt{43}}{1.6449}$$

$$\Rightarrow \sigma > 3069.7 \text{ (by GC)}.$$

Since $\sigma > 0$, squaring preserves the inequality:

$$\sigma^2 > 3069.7^2$$

$$\Rightarrow \sigma^2 > 9.42 \times 10^6 \text{ (3 s.f.)}.$$

To check your answer: at $\sigma^2 = 9.42 \times 10^6$ exactly, $z = \frac{-770}{\sqrt{9.42\times10^6/43}} = -1.645$, right at the critical value; a larger variance gives a z closer to 0, which is further from the critical region, consistent with not rejecting H_0. ✓

Note: there is no upper bound on σ^2 — as the variance used increases, the test statistic moves towards 0 and stays out of the (lower-tailed) critical region, so every larger variance is also consistent with not rejecting H_0.

$\therefore$ the variance used in calculating the test statistic satisfies $\sigma^2 > 9.42 \times 10^6 \text{ hours}^2$ (3 s.f.).

Question 6 7 marks

Probability · tree diagrams · paths

Part (i) 2 marks

Since there are 9 endpoints A–I, every journey from S to an endpoint passes through exactly 8 junctions, and at each junction the bug independently takes the left fork (probability p) or the right fork (probability q). Finishing at the k-th endpoint from the top (A being $k = 1$, I being $k = 9$) requires exactly $(k - 1)$ right forks and $(9 - k)$ left forks among these 8 junctions — moving one letter further down the row for each extra right fork taken.

D is the 4th endpoint, so reaching D requires exactly 3 right forks and 5 left forks. By independence of the junctions, any one such route has probability p^5q^3, and the number of such routes is the number of ways to choose which 3 of the 8 junctions are the right forks, $\binom{8}{3}$.

$$P(\text{bug finishes at D})$$

$$= \binom{8}{3} p^5 q^3$$

$$= 56p^5q^3. \quad \text{(shown)}$$

Part (ii) 4 marks

Let D_k be the probability that the bug finishes at the k-th endpoint from the top, so

$$P(D_k) = \binom{8}{k-1} p^{9-k} q^{k-1}, \qquad k = 1, \ldots, 9.$$

D is $k = 4$; its immediate neighbours are C ($k = 3$) and E ($k = 5$).

APPROACH

Comparing $P(D_{k+1})$ with $P(D_k)$ gives the ratio $\frac{P(D_{k+1})}{P(D_k)} = \frac{9-k}{k} \cdot \frac{q}{p}$, which is strictly decreasing in k. This means the sequence $P(D_1), \ldots, P(D_9)$ increases and then decreases (it has a single peak), so it is enough to check that $P(D)$ exceeds its two neighbours C and E — this then guarantees $P(D)$ exceeds every other endpoint's probability as well, without checking each one individually.

Comparing D with C:

$$\frac{P(D)}{P(C)} = \frac{56p^5q^3}{28p^6q^2} = \frac{2q}{p}.$$

We need $P(D) > P(C)$, i.e. $\frac{2q}{p} > 1$:

$$\begin{aligned} 2q &> p \\ \Rightarrow 2(1-p) &> p \\ \Rightarrow 2 &> 3p \\ \Rightarrow p &< \frac{2}{3}. \end{aligned}$$

Comparing D with E:

$$\frac{P(D)}{P(E)} = \frac{56p^5q^3}{70p^4q^4} = \frac{4p}{5q}.$$

We need $P(D) > P(E)$, i.e. $\frac{4p}{5q} > 1$:

$$\begin{aligned} 4p &> 5q \\ \Rightarrow 4p &> 5(1-p) \\ \Rightarrow 9p &> 5 \\ \Rightarrow p &> \frac{5}{9}. \end{aligned}$$

To check your answer: at $p = \frac{5}{9}$, $P(D) = P(E)$ exactly, and at $p = \frac{2}{3}$, $P(D) = P(C)$ exactly — the two boundary values recovered directly by setting each ratio equal to 1. ✓

∴ the possible range of values of p is $\frac{5}{9} < p < \frac{2}{3}$.

Part (iii) 1 mark

At each of the 8 junctions, independently of the other junctions, the bug either continues along a fork (probability $0.9p + 0.9q = 0.9(p+q) = 0.9$) or is swallowed by the black hole (probability 0.1). The bug reaches one of the endpoints A–I if and only if it survives all 8 junctions.

$$P(\text{reaches an endpoint}) = 0.9^8 = 0.430 \text{ (3 s.f., by GC)}.$$

Explanation: reaching an endpoint at all does not depend on which endpoint, only on never being swallowed, so the fork probabilities p and q do not appear in the answer — only their combined survival probability 0.9 at each junction matters.

$\therefore P(\text{bug reaches an endpoint without being swallowed}) = 0.430.$

Question 7 8 marks

Probability · independent and exclusive events

Part (i) 2 marks

Since A and B are independent, $P(A \cap B) = ab$.

$$\begin{aligned} & P(A' \cap B') \\ &= 1 - P(A \cup B) \\ &= 1 - [P(A) + P(B) - P(A \cap B)] \\ &= 1 - a - b + ab \\ &= (1-a)(1-b). \end{aligned}$$

Since $P(A') = 1 - a$ and $P(B') = 1 - b$, this says

$$P(A' \cap B') = P(A')\,P(B'),$$

so A' and B' are independent events. (shown)

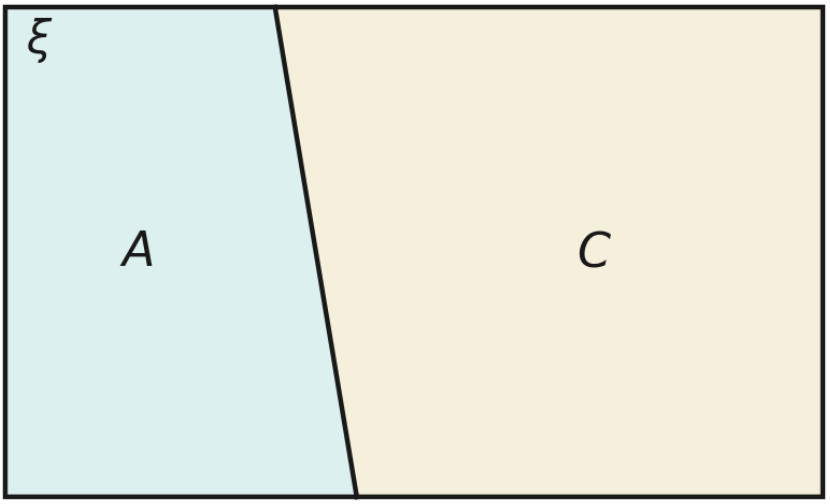
Part (ii) 2 marks

Since A and C are mutually exclusive, $P(A \cap C) = 0$.

$$\begin{aligned} & P(A' \cap C') \\ &= 1 - P(A \cup C) \\ &= 1 - [P(A) + P(C) - P(A \cap C)] \\ &= 1 - a - c. \end{aligned}$$

If A' and C' are also mutually exclusive, then $P(A' \cap C') = 0$, i.e. $a + c = 1$. Together with $A \cap C = \varnothing$, this means every outcome in the sample space lies in exactly one of A or C — the two events partition the universal set completely, with no overlap and no region left outside both.

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Part (iii) 4 marks

APPROACH

A and C are mutually exclusive, so A, C and $N := (A \cup C)'$ partition the sample space into three disjoint pieces. Splitting B across this partition, $P(B) = P(A \cap B) + P(C \cap B) + P(N \cap B)$. This turns the two given pieces of data — $P(B \cap C)$ and $P(A' \cap B' \cap C') = P(N \cap B')$ — together with A, B independence, into two equations relating $P(A \cap B)$ to $c = P(C)$, the one remaining unknown. The extreme values of $P(A \cap B)$ then come from the extreme values of c allowed by the constraints of the problem (each probability lying in $[0, 1]$).

We are given $a = P(A) = \frac{2}{5}$, $P(B \cap C) = \frac{1}{5}$ and $P(A' \cap B' \cap C') = \frac{1}{10}$. Write $c = P(C)$ and $b = P(B)$, both unknown for now, and let $N = (A \cup C)'$, so $P(N) = 1 - a - c = \frac{3}{5} - c$.

Since $A \cap C = \varnothing$, the events $A \cap B$, $C \cap B$ and $N \cap B$ partition B:

$$b = P(A \cap B) + P(C \cap B) + P(N \cap B).$$

Using A, B independence, $P(A \cap B) = ab = \frac{2}{5}b$. Writing $z = P(N \cap B)$,

$$b = \frac{2}{5}b + \frac{1}{5} + z.$$

Also, $P(A' \cap B' \cap C') = P(N \cap B') = P(N) - P(N \cap B) = \left(\frac{3}{5} - c\right) - z$, and this equals $\frac{1}{10}$:

$$\left(\frac{3}{5} - c\right) - z = \frac{1}{10}$$

$$\Rightarrow z = \frac{1}{2} - c.$$

Substituting into the equation for b:

$$b = \frac{2}{5}b + \frac{1}{5} + \left(\frac{1}{2} - c\right)$$

$$\Rightarrow \frac{3}{5}b = \frac{7}{10} - c$$

$$\Rightarrow b = \frac{7}{6} - \frac{5}{3}c.$$

Therefore

$$P(A \cap B) = \frac{2}{5}b = \frac{2}{5}\left(\frac{7}{6} - \frac{5}{3}c\right) = \frac{7}{15} - \frac{2}{3}c,$$

a decreasing function of c, so the maximum of $P(A \cap B)$ occurs at the smallest possible c, and the minimum at the largest possible c.

We now bound c. Since $B \cap C \subseteq C$, $c \geq P(B \cap C) = \frac{1}{5}$. Since $z = P(N \cap B)$ is a probability, $z \geq 0$, i.e. $c \leq \frac{1}{2}$. (Checking the remaining probabilities — b, $P(A \cap B)$, $z \leq P(N)$ — against these two bounds shows none of them cuts the range further, and $a + c \leq \frac{2}{5} + \frac{1}{2} < 1$ throughout, so A' and C' staying not mutually exclusive is automatic.)

$$\frac{1}{5} \leq c \leq \frac{1}{2}.$$

At $c = \frac{1}{5}$: $P(A \cap B) = \frac{7}{15} - \frac{2}{3} \cdot \frac{1}{5} = \frac{7}{15} - \frac{2}{15} = \frac{5}{15} = \frac{1}{3}$.

At $c = \frac{1}{2}$: $P(A \cap B) = \frac{7}{15} - \frac{2}{3} \cdot \frac{1}{2} = \frac{7}{15} - \frac{1}{3} = \frac{7}{15} - \frac{5}{15} = \frac{2}{15}$.

To check your answer: at $c = \frac{1}{5}$, $b = \frac{7}{6} - \frac{5}{3} \cdot \frac{1}{5} = \frac{5}{6}$, and $P(A \cap B) = \frac{1}{3} \leq \min(a, b) = \frac{2}{5}$, a valid probability; at $c = \frac{1}{2}$, $b = \frac{1}{3}$ and $P(A \cap B) = \frac{2}{15} \leq \frac{2}{5}$ also. ✓

$\therefore$ the maximum possible value of $P(A \cap B)$ is $\frac{1}{3}$, and the minimum possible value is $\frac{2}{15}$.

Question 8 11 marks

Discrete random variables · expectation and variance

Part (i) 4 marks

APPROACH

The bag has $n + 5$ balls in total: 2 numbered 3, 3 numbered 4, and n numbered 5. Drawing 2 balls without replacement, the possible values of S come from pairing these three labels: $3 + 3 = 6$, $3 + 4 = 7$, $3 + 5 = 8$ or $4 + 4 = 8$, $4 + 5 = 9$, $5 + 5 = 10$. We count the number of ways to get each sum and divide by the total number of ways to choose 2 balls from $n + 5$.

The total number of ways to choose 2 balls from the bag is

$$\binom{n+5}{2} = \frac{(n+5)(n+4)}{2}.$$

For each value of S:

- $S = 6$: both balls numbered 3, giving $\binom{2}{2} = 1$ way.
- $S = 7$: one numbered 3 and one numbered 4, giving $2 \times 3 = 6$ ways.
- $S = 8$: either both numbered 4 ($\binom{3}{2} = 3$ ways) or one numbered 3 and one numbered 5 ($2n$ ways), giving $2n + 3$ ways.
- $S = 9$: one numbered 4 and one numbered 5, giving $3n$ ways.

- $S = 10$: both numbered 5, giving $\binom{n}{2} = \frac{n(n-1)}{2}$ ways.

Dividing each by $\binom{n+5}{2}$ and simplifying to a common denominator $(n+5)(n+4)$,

s	6	7	8	9	10
$P(S=s)$	$\frac{2}{(n+4)(n+5)}$	$\frac{12}{(n+4)(n+5)}$	$\frac{4n+6}{(n+4)(n+5)}$	$\frac{6n}{(n+4)(n+5)}$	$\frac{n(n-1)}{(n+4)(n+5)}$

To check your answer: summing the five numerators gives $2 + 12 + (4n+6) + 6n + n(n-1) = n^2 + 9n + 20 = (n+4)(n+5)$, matching the common denominator exactly, so the probabilities sum to 1 for every n. ✓

Part (ii) 1 mark

With $n = 1$ there are 6 balls in the bag, with only **one** ball numbered 5. Substituting $n = 1$ into the formula for $P(S = 10)$ from **(i)**,

$$P(S=10) = \frac{1(1-1)}{(1+4)(1+5)} = \frac{0}{30} = 0.$$

Explanation: $S = 10$ can only arise from drawing two balls both numbered 5. Since there is only one ball numbered 5 in the bag when $n = 1$, it is impossible to draw two of them, so this outcome cannot occur.

$\therefore P(S = 10) = 0$, since the bag contains only one 5-numbered ball when $n = 1$, so two 5s cannot both be drawn.

Part (iii) 6 marks

APPROACH

We compute $E(S)$ and $E\left(S^2\right)$ directly from the distribution in (i), then use $Var(S) = E\left(S^2\right) - [E(S)]^2$ and combine everything over the common denominator $(n+4)(n+5)$ (squared, where needed) to match the required form.

Using the distribution from (i),

$$E(S) = \sum s\,P(S=s)$$
$$= \frac{6(2) + 7(12) + 8(4n+6) + 9(6n) + 10\,n(n-1)}{(n+4)(n+5)}$$
$$= \frac{10n^2 + 76n + 144}{(n+4)(n+5)}$$

The numerator has $(n+4)$ as a factor:

$$10n^2 + 76n + 144 = (n+4)(10n+36)$$
$$E(S) = \frac{(n+4)(10n+36)}{(n+4)(n+5)} = \frac{10n+36}{n+5}. \quad \text{(shown)}$$

For the variance, we first find $E\left(S^2\right)$:

$$E\left(S^2\right) = \sum s^2\,P(S=s)$$

$$= \frac{36(2) + 49(12) + 64(4n+6) + 81(6n) + 100\,n(n-1)}{(n+4)(n+5)}$$

$$= \frac{100n^2 + 642n + 1044}{(n+4)(n+5)}$$

$$Var(S) = E\left(S^2\right) - [E(S)]^2$$

$$= \frac{100n^2 + 642n + 1044}{(n+4)(n+5)} - \left(\frac{10n+36}{n+5}\right)^2$$

Writing both terms over the common denominator $(n+4)(n+5)^2$,

$$Var(S) = \frac{\left(100n^2 + 642n + 1044\right)(n+5) - (10n+36)^2(n+4)}{(n+4)(n+5)^2}$$

By GC, the numerator expands and simplifies to

$$\left(100n^2 + 642n + 1044\right)(n+5) - (10n+36)^2(n+4) = 22n^2 + 78n + 36 = 2(n+3)(11n+6)$$

$$Var(S) = \frac{22n^2 + 78n + 36}{(n+5)^2(n+4)}. \quad \text{(shown)}$$

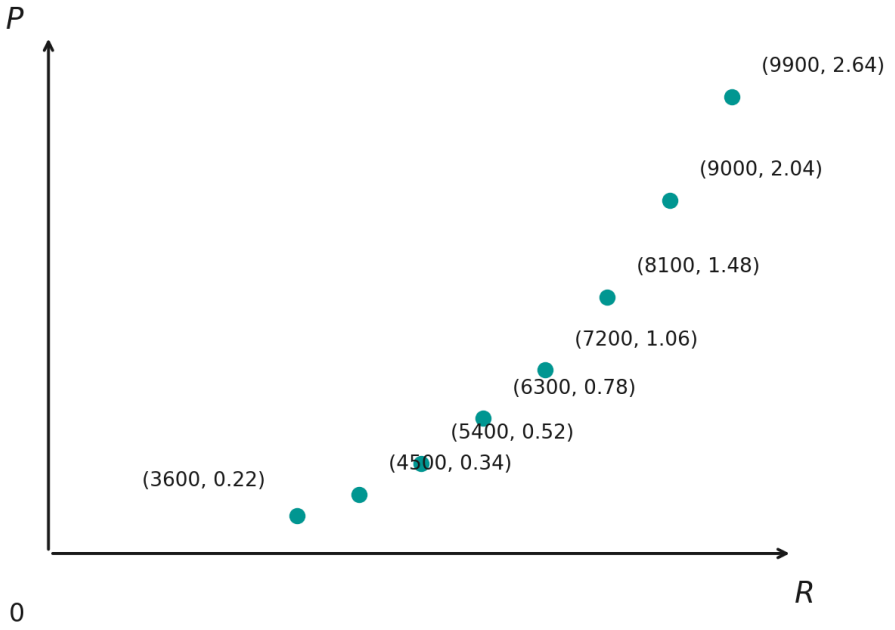
To check your answer: at $n = 1$ (the case in part **(ii)**), $E(S) = \frac{46}{6} = \frac{23}{3}$ and $Var(S) = \frac{136}{180} = \frac{34}{45}$; computing these directly from the six-ball distribution in (ii) by brute force gives the same values. ✓

$\therefore E(S) = \frac{10n+36}{n+5}$ and $Var(S) = \frac{g(n)}{(n+5)^2(n+4)}$, where $g(n) = 22n^2 + 78n + 36$.

Question 9 12 marks

Correlation and regression · regression lines

Part (i) 2 marks



The points do not lie close to a straight line — they curve upwards more and more steeply as R increases, bending away from any single straight line through them. This suggests the relationship between P and R is **not** well modelled by $P = aR + b$; a curved (non-linear) model fits the pattern of the data better.

Part (ii) 6 marks

> **APPROACH**
>
> to compare the two proposed models, $P = aR + b$ and $P = aR^2 + b$, we compute the product moment correlation coefficient (PMCC) for each — the first between P and R, the second between P and R^2 (since $P = aR^2 + b$ is linear in R^2, not in R) — and take the model whose r is closer to 1.

By GC,

$$r_{P,\,R} = 0.969 \text{ (3 s.f., by GC)}$$
$$r_{P,\,R^2} = 0.993 \text{ (3 s.f., by GC)}$$

Explanation: $r_{P,R^2} = 0.993$ is closer to 1 than $r_{P,R} = 0.969$, so $P = aR^2 + b$ gives a stronger linear relationship (between P and R^2) than $P = aR + b$ does (between P and R), and is therefore the better model.

By GC, the regression line of P on R^2 is

$$P = 2.85 \times 10^{-8} R^2 - 0.283 \text{ (3 s.f., by GC)}.$$

$\therefore P = aR^2 + b$ is the better model, with $a = 2.85 \times 10^{-8}$ and $b = -0.283$, both correct to 3 s.f.

Part (iii) 2 marks

Using the equation from (ii), we solve for R when $P = 0.9$:

$$\begin{aligned} 0.9 &= \left(2.845744\ldots \times 10^{-8}\right) R^2 - 0.282608\ldots \\ \Rightarrow R^2 &= \frac{0.9 + 0.282608\ldots}{2.845744\ldots \times 10^{-8}} \\ \Rightarrow R &= 6450 \text{ (3 s.f., by GC)}, \end{aligned}$$

taking the positive root since speed $R > 0$.

Explanation: $R = 6450$ lies within the range of the given data ($3600 \le R \le 9900$), so this is an interpolation, and the estimate is reliable.

$\therefore$ the estimated speed is $R = 6450$ rpm (3 s.f.), and this estimate is reliable.

Part (iv) 2 marks

Using the equation from (ii), at $R = 3300$,

$$P = \left(2.845744\ldots \times 10^{-8}\right)(3300)^2 - 0.282608\ldots$$

$$= 0.0273 \text{ (3 s.f., by GC)}.$$

Explanation: $R = 3300$ lies outside the range of the given data ($3600 \leq R \leq 9900$), so this is an extrapolation, and the estimate is unreliable — the equation is not known to hold below $R = 3600$.

$\therefore$ the estimated power used is $P = 0.0273$ watts (3 s.f.), but this estimate is unreliable since it extrapolates beyond the data.

Part (v)

APPROACH

the equation in (ii) uses R in revolutions per minute. If r denotes the speed in revolutions per second, then $R = 60r$ (there are 60 seconds in a minute), so we substitute $R = 60r$ into the equation from (ii).

Let r be the fan speed in revolutions per second, so $R = 60r$. Substituting into $P = aR^2 + b$ from **(ii)**,

$$P = a(60r)^2 + b$$
$$= 3600ar^2 + b$$

By GC, $3600a = 3600 \times 2.845744\ldots \times 10^{-8} = 1.02 \times 10^{-4}$ (3 s.f.).

$\therefore P = 1.02 \times 10^{-4}r^2 - 0.283$, correct to 3 s.f., where r is the fan speed in revolutions per second.

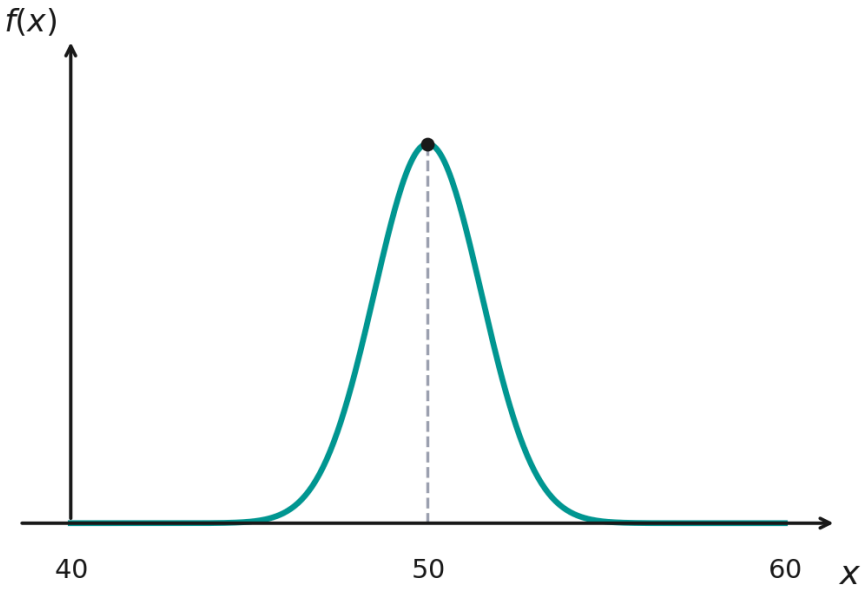
Question 10 15 marks

Normal distribution · sums of normal variables

Part (i) 2 marks

Let X be the mass, in grams, of a randomly chosen light bulb. Then $X \sim N\left(50,\ 1.5^2\right)$, so the density is a symmetric bell curve centred at $x = 50$.

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Part (ii) 1 mark

Standardising, with $Z = \frac{X-50}{1.5} \sim N(0,1)$,

$$\begin{aligned} & P(X < 50.4) \\ &= P\left(Z < \frac{50.4-50}{1.5}\right) \\ &= 0.605 \text{ (3 s.f., by GC)}. \end{aligned}$$

$\therefore P(X < 50.4) = 0.605$, correct to 3 s.f.

Part (iii) 2 marks

APPROACH

"total mass of 4 randomly chosen empty boxes" is a sum of 4 independent normal random variables, not 4 times a single box's mass — the mean scales by 4 but the variance also scales by 4 (not 16), since the boxes are independent.

Let Y be the mass, in grams, of a randomly chosen empty box, so $Y \sim N\left(75,\ 2^2\right)$. Let $T = Y_1 + Y_2 + Y_3 + Y_4$ be the total mass of 4 independently, randomly chosen empty boxes. Since the Y_i are independent,

$$T \sim N\left(4(75),\ 4\left(2^2\right)\right) = N(300,\ 16).$$

$$P(T > 297) = 0.773 \text{ (3 s.f., by GC)}.$$

$\therefore P(T > 297) = 0.773$, correct to 3 s.f.

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Part (iv) 3 marks

Let $W = X + Y$ be the total mass, in grams, of a randomly chosen light bulb together with a randomly chosen box. Assuming the bulb's mass and the box's mass are independent,

$$W \sim N\left(50 + 75,\ 1.5^2 + 2^2\right) = N(125,\ 6.25).$$
$$P(124.9 < W < 125.7) = 0.126\ (3\text{ s.f., by GC}).$$

$\therefore P(124.9 < W < 125.7) = 0.126$, correct to 3 s.f.

Part (v) 4 marks

APPROACH

the padding's mass is 30% of the bulb's mass, i.e. $0.3X$ — a fixed multiple of X, not a separate independent random variable — so the total mass of box, bulb and padding is $Y + X + 0.3X = Y + 1.3X$. We find the distribution of $Y + 1.3X$, then use the given tail probability to solve for k.

Let $U = Y + X + 0.3X = Y + 1.3X$ be the total mass, in grams, of a box containing a bulb and its padding. Since X and Y are independent,

$$1.3X \sim N\left(1.3(50),\ 1.3^2\left(1.5^2\right)\right) = N(65,\ 3.8025)$$
$$U = Y + 1.3X \sim N\left(75 + 65,\ 2^2 + 3.8025\right) = N(140,\ 7.8025).$$

We are given $P(U > k) = 0.9$. Standardising,

$$P\left(Z > \frac{k - 140}{\sqrt{7.8025}}\right) = 0.9,\ \text{where } Z \sim N(0, 1).$$

By GC (inverse normal),

$$\frac{k - 140}{\sqrt{7.8025}} = -1.2816\ldots$$
$$\Rightarrow k = 140 - 1.2816\ldots \times \sqrt{7.8025}$$
$$= 136\ (3\text{ s.f.})$$

To check your answer: since $P(U > k) = 0.9 > 0.5$, k should be below the mean 140; $k = 136$ is indeed below 140. ✓

$\therefore k = 136$, correct to 3 s.f.

Part (vi) 3 marks

APPROACH

as in (iii), "total mass of 4 boxes, each containing a bulb and padding" sums 4 independent copies of U from (v).

Let $S = U_1 + U_2 + U_3 + U_4$ be the total mass of 4 independently, randomly chosen boxes, each containing a bulb and padding. Since the U_i are independent,

$$S \sim N\left(4(140),\ 4(7.8025)\right) = N(560,\ 31.21).$$

$$P(S > 565) = 0.185 \text{ (3 s.f., by GC)}.$$

$\therefore P(S > 565) = 0.185$, correct to 3 s.f.

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