TEN-YEAR SERIES · WORKED SOLUTIONS

2018 A-Level H2 Mathematics Paper 1

11 questions · 100 marks

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Question 1 6 marks

Differentiation · integration · definite integrals

Part (i) 2 marks

We use the quotient rule with $u = \ln x$ and $v = x$, so $u' = \frac{1}{x}$ and $v' = 1$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{vu' - uv'}{v^2} \\ &= \frac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2} \\ &= \frac{1 - \ln x}{x^2}\end{aligned}$$

$\therefore \frac{dy}{dx} = \frac{1-\ln x}{x^2}$.

Part (ii) 4 marks

APPROACH

The integrand $\frac{\ln x}{x^2}$ is not itself $\frac{dy}{dx}$, but comparing the two shows how they relate: split $\frac{dy}{dx}$ from (i) into two fractions and isolate the term we want.

Since

$$\frac{dy}{dx} = \frac{1}{x^2} - \frac{\ln x}{x^2},$$

rearranging gives

$$\frac{\ln x}{x^2} = \frac{1}{x^2} - \frac{dy}{dx}.$$

Integrating both sides from 1 to e:

$$\begin{aligned}&\int_1^e \frac{\ln x}{x^2}\,dx \\ &= \int_1^e \frac{1}{x^2}\,dx - \int_1^e \frac{dy}{dx}\,dx \\ &= \left[-\frac{1}{x}\right]_1^e - [y]_1^e\end{aligned}$$

Since $y = \frac{\ln x}{x}$, we have $y(e) = \frac{1}{e}$ and $y(1) = 0$.

$$\begin{aligned}&\int_1^e \frac{\ln x}{x^2}\,dx \\ &= \left(-\frac{1}{e} + 1\right) - \left(\frac{1}{e} - 0\right) \\ &= 1 - \frac{2}{e}\end{aligned}$$

To check your answer: By GC, $\int_1^e \frac{\ln x}{x^2}\,dx \approx 0.264$, and $1 - \frac{2}{e} \approx 0.264$. ✓

$\therefore \int_1^e \frac{\ln x}{x^2}\,dx = 1 - \frac{2}{e}$.

Question 2 6 marks

Integration · area between a curve and a line

Part (i) 2 marks

Since A and B lie on both the curve and the line, we substitute $y = \frac{3}{x}$ into $y + 2x = 7$.

$$\frac{3}{x} + 2x = 7$$
$$3 + 2x^2 = 7x$$
$$2x^2 - 7x + 3 = 0$$
$$(2x - 1)(x - 3) = 0$$

$\therefore$ the x-coordinates of A and B are $x = \frac{1}{2}$ and $x = 3$.

Part (ii) 4 marks

APPROACH

The region bounded by the curve and the line lies between $x = \frac{1}{2}$ and $x = 3$. Since the two boundaries are a curve and a line (not one curve and the x-axis), the volume of revolution is the volume swept by the line minus the volume swept by the curve, not a single $\pi \int y^2\,dx$.

Writing the line as $y = 7 - 2x$, we check which boundary is on top: at $x = 1$, the line gives $y = 5$ and the curve gives $y = 3$, so the line lies above the curve on $\frac{1}{2} < x < 3$. The required volume is therefore

$$V = \pi \int_{1/2}^{3} \left[(7 - 2x)^2 - \left(\frac{3}{x}\right)^2\right] dx.$$

Expanding the integrand,

$$V = \pi \int_{1/2}^{3} \left(4x^2 - 28x + 49 - \frac{9}{x^2}\right) dx$$
$$= \pi \left[\frac{4x^3}{3} - 14x^2 + 49x + \frac{9}{x}\right]_{1/2}^{3}$$

At $x = 3$:

$$\frac{4(3)^3}{3} - 14(3)^2 + 49(3) + \frac{9}{3} = 36 - 126 + 147 + 3 = 60$$

At $x = \frac{1}{2}$:

$$\frac{4\left(\frac{1}{2}\right)^3}{3} - 14\left(\frac{1}{2}\right)^2 + 49\left(\frac{1}{2}\right) + \frac{9}{\frac{1}{2}} = \frac{1}{6} - \frac{7}{2} + \frac{49}{2} + 18 = \frac{235}{6}$$

So

$$\begin{aligned} V &= \pi\left(60 - \frac{235}{6}\right) \\ &= \frac{125\pi}{6} \end{aligned}$$

$\therefore$ the exact volume generated is $\frac{125\pi}{6}$ units^3.

Question 3 7 marks

Differential equations · substitution

Part (i) 3 marks

Since $y = ux^2$, where u is a function of x, we differentiate using the product rule:

$$\frac{dy}{dx} = x^2\frac{du}{dx} + 2xu.$$

Substituting $y = ux^2$ and this expression for $\frac{dy}{dx}$ into $x\frac{dy}{dx} = 2y - 6$:

$$\begin{aligned} x\left(x^2\frac{du}{dx} + 2xu\right) &= 2\left(ux^2\right) - 6 \\ x^3\frac{du}{dx} + 2x^2u &= 2x^2u - 6 \\ x^3\frac{du}{dx} &= -6 \\ \frac{du}{dx} &= -\frac{6}{x^3} \quad \text{(shown)} \end{aligned}$$

$\therefore f(x) = -\frac{6}{x^3}$.

Part (ii) 4 marks

APPROACH

Having reduced the equation to $\frac{du}{dx} = f(x)$ in (i), we integrate directly to find u, then convert back to y using $y = ux^2$ and apply the given condition.

Integrating $\frac{du}{dx} = -6x^{-3}$ with respect to x:

$$\begin{aligned} u &= \int -6x^{-3}\,dx \\ &= 3x^{-2} + C \\ &= \frac{3}{x^2} + C, \end{aligned}$$

where C is an arbitrary constant. Since $y = ux^2$,

$$y = \left(\frac{3}{x^2} + C\right)x^2$$

$$= 3 + Cx^2.$$

Using the given condition $y = 2$ when $x = 1$:

$$2 = 3 + C$$
$$C = -1$$

$\therefore y = 3 - x^2$.

To check your answer: with $y = 3 - x^2$, $\frac{dy}{dx} = -2x$, so $x\frac{dy}{dx} = -2x^2$, and $2y - 6 = 2\left(3 - x^2\right) - 6 = -2x^2$. The two sides agree, and $y(1) = 3 - 1 = 2$ matches the given condition. ✓

Question 4 8 marks

Inequalities · modulus equations and inequalities

Part (i) 4 marks

APPROACH

An equation of the form $|f(x)| = g(x)$ splits into two cases according to the sign of $f(x)$: where $f(x) \geq 0$ we solve $f(x) = g(x)$, and where $f(x) < 0$ we solve $-f(x) = g(x)$. We first find where $2x^2 + 3x - 2$ changes sign.

Since $2x^2 + 3x - 2 = (2x - 1)(x + 2)$, this is zero at $x = -2$ and $x = \frac{1}{2}$, so $2x^2 + 3x - 2 \geq 0$ for $x \leq -2$ or $x \geq \frac{1}{2}$, and $2x^2 + 3x - 2 < 0$ for $-2 < x < \frac{1}{2}$.

Case 1: $x \leq -2$ **or** $x \geq \frac{1}{2}$. Here $\left|2x^2 + 3x - 2\right| = 2x^2 + 3x - 2$, so

$$2x^2 + 3x - 2 = 2 - x$$
$$2x^2 + 4x - 4 = 0$$
$$x^2 + 2x - 2 = 0$$
$$x = \frac{-2 \pm \sqrt{4 + 8}}{2} = -1 \pm \sqrt{3}$$

Since $-1 - \sqrt{3} \approx -2.73 \leq -2$ and $-1 + \sqrt{3} \approx 0.73 \geq \frac{1}{2}$, both roots lie in this case.

Case 2: $-2 < x < \frac{1}{2}$. Here $\left|2x^2 + 3x - 2\right| = -\left(2x^2 + 3x - 2\right)$, so

$$-\left(2x^2 + 3x - 2\right) = 2 - x$$
$$-2x^2 - 3x + 2 = 2 - x$$
$$-2x^2 - 2x = 0$$
$$-2x(x + 1) = 0$$

giving $x = 0$ (in range) or $x = -1$ (in range); both lie in $-2 < x < \frac{1}{2}$.

(All four values also satisfy $2 - x \geq 0$, needed since the left side is non-negative.)

$\therefore$ the exact roots are $x = -1 - \sqrt{3},\ -1,\ 0,\ -1 + \sqrt{3}$.

Part (ii) 4 marks

APPROACH

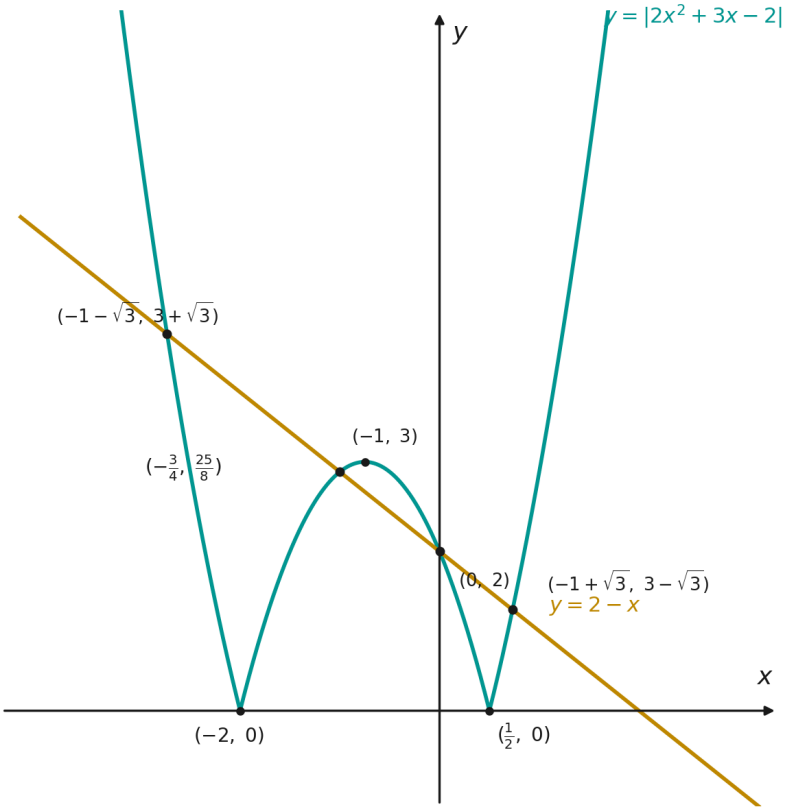
The graph of $y = |2x^2 + 3x - 2|$ is the parabola $y = 2x^2 + 3x - 2$ reflected upward wherever it dips below the x-axis, i.e. on $-2 < x < \frac{1}{2}$, and left unchanged elsewhere. The reflected piece has the same vertex x-coordinate as the original parabola.

The original parabola $2x^2 + 3x - 2$ has vertex at $x = -\frac{3}{4}$, with value $2\left(-\frac{3}{4}\right)^2 + 3\left(-\frac{3}{4}\right) - 2 = -\frac{25}{8}$; after reflection, the middle hump of $y = |2x^2 + 3x - 2|$ has a maximum at $\left(-\frac{3}{4},\ \frac{25}{8}\right)$. The graph touches the x-axis (corners, not smooth turning points) at $(-2, 0)$ and $\left(\frac{1}{2}, 0\right)$, and its y-intercept is $(0, 2)$.

The line $y = 2 - x$ has intercepts $(0, 2)$ and $(2, 0)$.

Using the roots from (i), the graphs meet where $y = 2 - x$, at

$$\left(-1 - \sqrt{3},\ 3 + \sqrt{3}\right), \quad (-1,\ 3), \quad (0,\ 2), \quad \left(-1 + \sqrt{3},\ 3 - \sqrt{3}\right).$$



Hence: the inequality $|2x^2 + 3x - 2| < 2 - x$ holds exactly where the curve lies strictly below the line. Reading this off the graph, the curve dips below the line on two intervals between consecutive intersections: from $-1 - \sqrt{3}$ to -1, and from 0 to $-1 + \sqrt{3}$ (checking, e.g., $x = -2$ gives curve $= 0 < 4 =$ line, and $x = 0.3$ gives curve $\approx 0.92 < 1.7 \approx$ line, confirming both intervals; on $-1 < x < 0$ the curve sits above the line, e.g. at $x = -0.5$, curve $= 3 > 2.5 =$ line).

$\therefore -1 - \sqrt{3} < x < -1$ or $0 < x < -1 + \sqrt{3}$.

Question 5 5 marks

Functions · composite functions

APPROACH

The condition $ff = g$ means $f\left(f(x)\right) = x$ for every x in the domain, i.e. f is a self-inverse function. Rather than guessing, we compute $f\left(f(x)\right)$ explicitly and force it to equal x identically; matching coefficients then pins down b. Since f turns out to be self-inverse, f^{-1} falls out for free once b is known.

Since $f : x \mapsto \frac{x+a}{x+b}$,

$$\begin{aligned} & f\left(f(x)\right) \\ &= \frac{\frac{x+a}{x+b} + a}{\frac{x+a}{x+b} + b} \\ &= \frac{(x+a) + a(x+b)}{(x+a) + b(x+b)} \\ &= \frac{(1+a)x + a(1+b)}{(1+b)x + a + b^2} \end{aligned}$$

For $ff = g$, this must equal x for every x in the domain, so

$$(1+a)x + a(1+b) = x\left[(1+b)x + a + b^2\right].$$

Comparing coefficients of x^2 on both sides forces $1 + b = 0$, i.e. $b = -1$.

(Checking the remaining coefficients with $b = -1$: the x^1 terms give $1 + a = a + b^2 = a + 1$, true for all a; the constant term gives $a(1+b) = 0 = 0$, also automatic. So $b = -1$ is the only requirement, consistent with $a \neq -1$ being given separately to keep f well-defined and non-constant.)

$\therefore b = -1$.

With $b = -1$, $f : x \mapsto \frac{x+a}{x-1}$ for $x \in \mathbb{R}$, $x \neq 1$.

Note: Since $ff = g$ means f is its own inverse, we expect $f^{-1}(x) = f(x)$; we find f^{-1} directly to confirm this.

Let $y = \frac{x+a}{x-1}$. Then

$$\begin{aligned} y(x-1) &= x + a \\ yx - y &= x + a \\ x(y-1) &= a + y \\ x &= \frac{a+y}{y-1}. \end{aligned}$$

Interchanging x and y,

$$f^{-1}(x) = \frac{x+a}{x-1}.$$

To check your answer: $f^{-1}(x) = \frac{x+a}{x-1} = f(x)$, matching the self-inverse property forced by $ff = g$. ✓

$\therefore f^{-1}(x) = \frac{x+a}{x-1}$.

Question 6 7 marks

Vectors · vector product

Part (i) 2 marks

Since $\mathbf{a} \times 3\mathbf{b} = 2\mathbf{a} \times \mathbf{c}$,

$$3(\mathbf{a} \times \mathbf{b}) = 2(\mathbf{a} \times \mathbf{c})$$
$$3(\mathbf{a} \times \mathbf{b}) - 2(\mathbf{a} \times \mathbf{c}) = \mathbf{0}$$
$$\mathbf{a} \times (3\mathbf{b} - 2\mathbf{c}) = \mathbf{0}.$$

Since $\mathbf{a} \neq \mathbf{0}$, a vector cross product $\mathbf{a} \times \mathbf{v} = \mathbf{0}$ means $\mathbf{v}$ is parallel to $\mathbf{a}$ (or $\mathbf{v} = \mathbf{0}$, the degenerate case of parallel). So $3\mathbf{b} - 2\mathbf{c}$ is parallel to $\mathbf{a}$, i.e.

$$3\mathbf{b} - 2\mathbf{c} = \lambda\mathbf{a}$$

for some constant λ. (shown)

Part (ii) 5 marks

APPROACH

We are given magnitudes and an angle, which is scalar-product information, so we turn the vector equation from (i) into a scalar one by taking the modulus of both sides and squaring — this brings in $|\mathbf{b}|$, $|\mathbf{c}|$ and $\mathbf{b} \cdot \mathbf{c}$, all of which we know, and $|\mathbf{a}| = 1$.

Since $\mathbf{a}$ and $\mathbf{c}$ are unit vectors, $|\mathbf{a}| = 1$ and $|\mathbf{c}| = 1$; also $|\mathbf{b}| = 4$ and the angle between $\mathbf{b}$ and $\mathbf{c}$ is $60°$, so

$$\mathbf{b} \cdot \mathbf{c} = |\mathbf{b}|\,|\mathbf{c}| \cos 60° = 4(1)\left(\frac{1}{2}\right) = 2.$$

Taking the modulus squared of both sides of $3\mathbf{b} - 2\mathbf{c} = \lambda\mathbf{a}$,

$$|3\mathbf{b} - 2\mathbf{c}|^2 = \lambda^2 |\mathbf{a}|^2$$
$$(3\mathbf{b} - 2\mathbf{c}) \cdot (3\mathbf{b} - 2\mathbf{c}) = \lambda^2$$
$$9|\mathbf{b}|^2 - 12(\mathbf{b} \cdot \mathbf{c}) + 4|\mathbf{c}|^2 = \lambda^2$$
$$9(16) - 12(2) + 4(1) = \lambda^2$$
$$124 = \lambda^2.$$

Solving,

$$\lambda = \pm\sqrt{124} = \pm 2\sqrt{31}.$$

$\therefore \lambda = 2\sqrt{31}$ or $\lambda = -2\sqrt{31}$.

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Question 7 9 marks

Differentiation · implicit · tangents

Part (i) 3 marks

APPROACH

The equation for C has y terms buried inside a fraction, which is awkward to differentiate directly. We first clear denominators to get a polynomial relation between x and y, then differentiate implicitly.

Since $\frac{x^2-4y^2}{x^2+xy^2} = \frac{1}{2}$, cross-multiplying,

$$\begin{aligned} 2\left(x^2-4y^2\right) &= x^2+xy^2 \\ 2x^2-8y^2 &= x^2+xy^2 \\ x^2-8y^2-xy^2 &= 0 \\ x^2 &= y^2(x+8). \end{aligned}$$

Differentiating both sides with respect to x, using the product rule on the right (treating y as a function of x),

$$\begin{aligned} 2x &= 2y\frac{dy}{dx}(x+8)+y^2 \\ 2y(x+8)\frac{dy}{dx} &= 2x-y^2 \\ \frac{dy}{dx} &= \frac{2x-y^2}{2y(x+8)}. \end{aligned}$$

Since $2y(x+8) = 2xy+16y$,

$$\frac{dy}{dx} = \frac{2x-y^2}{2xy+16y}. \quad \text{(shown)}$$

Part (ii) 6 marks

APPROACH

With $x=1$ fixed, the relation $x^2 = y^2(x+8)$ from (i) gives the two y-values of P and Q directly. We then find the gradient at each point from the given derivative, write down both tangents, and solve them simultaneously for N.

Since $x=1$ lies on C, substituting into $x^2 = y^2(x+8)$,

$$\begin{aligned} 1 &= y^2(9) \\ y^2 &= \frac{1}{9} \\ y &= \pm\frac{1}{3}. \end{aligned}$$

Take $P = \left(1,\ \frac{1}{3}\right)$ and $Q = \left(1,\ -\frac{1}{3}\right)$.

At P, using the result of (i) with $x=1$, $y=\frac{1}{3}$,

$$\left.\frac{dy}{dx}\right|_P = \frac{2(1) - \left(\frac{1}{3}\right)^2}{2(1)\left(\frac{1}{3}\right) + 16\left(\frac{1}{3}\right)} = \frac{\frac{17}{9}}{6} = \frac{17}{54}.$$

At Q, with $x = 1$, $y = -\frac{1}{3}$, the numerator is unchanged (it depends only on y^2) while the denominator changes sign, so

$$\left.\frac{dy}{dx}\right|_Q = -\frac{17}{54}.$$

The tangent to C at P is

$$y - \frac{1}{3} = \frac{17}{54}(x - 1),$$

and the tangent to C at Q is

$$y + \frac{1}{3} = -\frac{17}{54}(x - 1).$$

Note: Since these two tangents have equal and opposite gradients and P, Q are reflections of each other in the x-axis, adding the two equations eliminates the x-terms and shows N lies on the x-axis; we solve directly to confirm this and find the x-coordinate.

Solving the two tangent equations simultaneously: from the first, $y = \frac{1}{3} + \frac{17}{54}(x - 1)$; substituting into the second,

$$\frac{1}{3} + \frac{17}{54}(x - 1) + \frac{1}{3} = -\frac{17}{54}(x - 1)$$
$$\frac{34}{54}(x - 1) = -\frac{2}{3}$$
$$x - 1 = -\frac{2}{3} \cdot \frac{54}{34} = -\frac{18}{17}$$
$$x = -\frac{1}{17}.$$

Substituting back,

$$y = \frac{1}{3} + \frac{17}{54}\left(-\frac{18}{17}\right) = \frac{1}{3} - \frac{1}{3} = 0.$$

To check your answer: $N = \left(-\frac{1}{17},\ 0\right)$ lies on the x-axis, as expected from the symmetry of C (unchanged under $y \mapsto -y$) between the mirror-image points P and Q. ✓

$\therefore N = \left(-\frac{1}{17},\ 0\right).$

Question 8 10 marks

Sequences and series · recurrence relations

Part (i) 2 marks

Since $u_{n+1} = 2u_n + An$, taking $n = 1$ with $u_1 = 5$ and $u_2 = 15$,

$$u_2 = 2u_1 + A(1)$$
$$15 = 10 + A$$

$$A = 5.$$

Taking $n = 2$,

$$u_3 = 2u_2 + A(2) = 2(15) + 5(2) = 40.$$

$\therefore A = 5$ and $u_3 = 40$.

Part (ii) 4 marks

APPROACH

We are told $u_n = a\left(2^n\right) + bn + c$ satisfies the same recurrence as $u_{n+1} = 2u_n + 5n$ (using $A = 5$ from (i)). Substituting this form into the recurrence and comparing coefficients of n on both sides — since the equation must hold for every n — gives b and c; a is then found from the initial condition $u_1 = 5$, since the recurrence alone cannot fix a (any multiple of the solution of the homogeneous equation $u_{n+1} = 2u_n$ can be added).

Substituting $u_n = a\left(2^n\right) + bn + c$ into $u_{n+1} = 2u_n + 5n$,

$$a\left(2^{n+1}\right) + b(n+1) + c = 2\left[a\left(2^n\right) + bn + c\right] + 5n$$
$$a\left(2^{n+1}\right) + bn + b + c = a\left(2^{n+1}\right) + 2bn + 2c + 5n$$
$$bn + b + c = (2b+5)n + 2c$$
$$0 = (b+5)n + (c-b).$$

Since this holds for every n, both coefficients vanish:

$$b + 5 = 0 \Longrightarrow b = -5,$$
$$c - b = 0 \Longrightarrow c = b = -5.$$

Using $u_1 = 5$ with $b = -5$, $c = -5$,

$$u_1 = a(2) + b(1) + c$$
$$5 = 2a - 5 - 5$$
$$2a = 15$$
$$a = \frac{15}{2}.$$

To check your answer: with $a = \frac{15}{2}$, $b = -5$, $c = -5$: $u_2 = \frac{15}{2}(4) - 10 - 5 = 15$ and $u_3 = \frac{15}{2}(8) - 15 - 5 = 40$, matching (i). ✓

$\therefore a = \frac{15}{2}$, $b = -5$, $c = -5$.

Part (iii) 4 marks

Since $u_r = \frac{15}{2}\left(2^r\right) - 5r - 5$,

$$\sum_{r=1}^{n} u_r = \frac{15}{2}\sum_{r=1}^{n} 2^r - 5\sum_{r=1}^{n} r - 5\sum_{r=1}^{n} 1.$$

By GC (or the standard GP and AP sum formulas), $\sum_{r=1}^{n} 2^r = \frac{2(2^n-1)}{2-1} = 2^{n+1} - 2$ and $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$, so

$$\sum_{r=1}^{n} u_r = \frac{15}{2}\left(2^{n+1} - 2\right) - 5 \cdot \frac{n(n+1)}{2} - 5n.$$

$\therefore \sum_{r=1}^{n} u_r = \frac{15}{2}\left(2^{n+1} - 2\right) - \frac{5n(n+1)}{2} - 5n.$

Question 9 13 marks

Parametric equations · normals · area

Part (i) 4 marks

Since $x = 2\theta - \sin 2\theta$,

$$\begin{aligned}\frac{dx}{d\theta} &= 2 - 2\cos 2\theta \\ &= 2\left(1 - \cos 2\theta\right) \\ &= 2\left(2\sin^2\theta\right) \\ &= 4\sin^2\theta,\end{aligned}$$

using $1 - \cos 2\theta = 2\sin^2\theta$. Since $y = 2\sin^2\theta$,

$$\begin{aligned}\frac{dy}{d\theta} &= 4\sin\theta\cos\theta \\ &= 2\sin 2\theta.\end{aligned}$$

Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} \\ &= \frac{4\sin\theta\cos\theta}{4\sin^2\theta} \\ &= \frac{\cos\theta}{\sin\theta} \\ &= \cot\theta \quad \text{(shown)}\end{aligned}$$

Part (ii) 4 marks

> **APPROACH**
>
> The normal at $\theta = \alpha$ has gradient $-\tan\alpha$, the negative reciprocal of the gradient $\cot\alpha$ found in (i). Writing its equation and setting $y = 0$ gives the x-coordinate of A directly in terms of α; the point of the question is that the $\sin 2\alpha$ terms cancel, leaving the clean multiple $k\alpha$.

At $\theta = \alpha$, the point on C is $\left(2\alpha - \sin 2\alpha,\ 2\sin^2\alpha\right)$, and by (i) the gradient of C there is $\cot\alpha$. Since the normal is perpendicular to the curve, its gradient is $-\tan\alpha$. The normal is

$$y - 2\sin^2\alpha = -\tan\alpha\left[x - (2\alpha - \sin 2\alpha)\right].$$

Setting $y = 0$ to find where the normal meets the x-axis,

$$-2\sin^2\alpha = -\tan\alpha\,[x - (2\alpha - \sin 2\alpha)]$$
$$2\sin^2\alpha = \tan\alpha\,[x - (2\alpha - \sin 2\alpha)]$$
$$x - (2\alpha - \sin 2\alpha) = 2\sin^2\alpha\cot\alpha$$
$$x - (2\alpha - \sin 2\alpha) = 2\sin\alpha\cos\alpha$$
$$x - (2\alpha - \sin 2\alpha) = \sin 2\alpha.$$

Therefore

$$x = 2\alpha - \sin 2\alpha + \sin 2\alpha$$
$$x = 2\alpha.$$

To check your answer: the $\sin 2\alpha$ terms cancel exactly, leaving a linear multiple of α alone, consistent with the question stating the answer has the form $k\alpha$. ✓

$\therefore$ the x-coordinate of A is 2α, so $k = 2$.

Part (iii) 5 marks

APPROACH

The arc-length integrand looks unpleasant until $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$ is expanded: both terms carry a common factor of $16\sin^2\theta$, and $\sin^2\theta + \cos^2\theta = 1$ collapses the sum under the square root to a single power of $\sin\theta$, which integrates directly — no calculator needed.

From (i), $\frac{dx}{d\theta} = 4\sin^2\theta$ and $\frac{dy}{d\theta} = 4\sin\theta\cos\theta$. Then

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$$
$$= 16\sin^4\theta + 16\sin^2\theta\cos^2\theta$$
$$= 16\sin^2\theta\left(\sin^2\theta + \cos^2\theta\right)$$
$$= 16\sin^2\theta.$$

Since $0 \le \theta \le \pi$, $\sin\theta \ge 0$, so $\sqrt{16\sin^2\theta} = 4\sin\theta$. The total length of C is

$$\int_0^{\pi} 4\sin\theta\,d\theta$$
$$= [-4\cos\theta]_0^{\pi}$$
$$= (-4\cos\pi) - (-4\cos 0)$$
$$= 4 + 4$$
$$= 8.$$

$\therefore$ the total length of C is 8 units.

Question 10 13 marks

Differential equations · electrical circuit

Part (i) 2 marks

APPROACH

Differentiating the given equation with respect to t turns $\frac{q}{C}$ into $\frac{1}{C}\frac{dq}{dt} = \frac{I}{C}$ (using $I = \frac{dq}{dt}$) and turns V into $\frac{dV}{dt}$. The target equation has no term on the right, so this only works if $\frac{dV}{dt} = 0$ — the condition to state is that V is constant.

Since $L\frac{dI}{dt} + RI + \frac{q}{C} = V$, differentiating both sides with respect to t,

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}\frac{dq}{dt} = \frac{dV}{dt}.$$

Using $I = \frac{dq}{dt}$,

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = \frac{dV}{dt}.$$

Provided V is constant (does not vary with time), $\frac{dV}{dt} = 0$, so

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = 0 \quad \text{(shown)}$$

Part (ii) 5 marks

APPROACH

Substitute $I = \mathrm{Ate}^{-\frac{Rt}{2L}}$ into the differential equation of (i) and collect the result as a polynomial in t multiplying $e^{-\frac{Rt}{2L}}$. Since $e^{-\frac{Rt}{2L}} \neq 0$, this polynomial must vanish for every t, so each of its coefficients (the constant term and the coefficient of t) is separately zero; the coefficient of t is the equation that gives C.

Let $k = \frac{R}{2L}$, so $I = \mathrm{Ate}^{-kt}$. Then

$$\begin{aligned}\frac{dI}{dt} &= Ae^{-kt} - \mathrm{Akte}^{-kt} \\ &= Ae^{-kt}(1 - kt),\end{aligned}$$

and

$$\begin{aligned}\frac{d^2I}{dt^2} &= -\mathrm{kAe}^{-kt}(1 - kt) + Ae^{-kt}(-k) \\ &= Ae^{-kt}\left[-k(1 - kt) - k\right] \\ &= Ae^{-kt}\left(k^2t - 2k\right).\end{aligned}$$

Substituting into $L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = 0$,

$$\mathrm{LAe}^{-kt}\left(k^2t - 2k\right) + \mathrm{RAe}^{-kt}(1 - kt) + \frac{1}{C}\mathrm{Ate}^{-kt} = 0.$$

Since $A \neq 0$ and $e^{-kt} \neq 0$, we can divide throughout by Ae^{-kt}:

$$L\left(k^2t - 2k\right) + R(1 - kt) + \frac{t}{C} = 0$$

Collecting this as a polynomial in t,

$$\left(Lk^2 - Rk + \frac{1}{C}\right)t + (R - 2Lk) = 0.$$

This must hold for every t, so both the constant term and the coefficient of t are zero. Since $k = \frac{R}{2L}$, the constant term $R - 2Lk = R - 2L \cdot \frac{R}{2L} = 0$ automatically. The coefficient of t then gives

$$Lk^2 - Rk + \frac{1}{C} = 0.$$

Substituting $k = \frac{R}{2L}$,

$$L\left(\frac{R}{2L}\right)^2 - R\left(\frac{R}{2L}\right) + \frac{1}{C} = 0$$
$$\frac{R^2}{4L} - \frac{R^2}{2L} + \frac{1}{C} = 0$$
$$-\frac{R^2}{4L} + \frac{1}{C} = 0$$
$$\frac{1}{C} = \frac{R^2}{4L}.$$

$\therefore C = \frac{4L}{R^2}$ (shown)

Part (iii) 4 marks

Since $R = 4$, $L = 3$, $C = 0.75$, we check $\frac{4L}{R^2} = \frac{4(3)}{16} = 0.75$, consistent with (ii), and

$$k = \frac{R}{2L} = \frac{4}{6} = \frac{2}{3}.$$

So $I = \mathrm{Ate}^{-\frac{2}{3}t}$, with derivative (from (ii), with this k)

$$\frac{dI}{dt} = Ae^{-\frac{2}{3}t}\left(1 - \frac{2}{3}t\right).$$

At a stationary value, $\frac{dI}{dt} = 0$. Since $Ae^{-\frac{2}{3}t} \neq 0$ (as $A > 0$),

$$1 - \frac{2}{3}t = 0$$
$$t = \frac{3}{2}.$$

The corresponding value of I is

$$I = A\left(\frac{3}{2}\right)e^{-\frac{2}{3}\left(\frac{3}{2}\right)}$$
$$= \frac{3A}{2}e^{-1}.$$

To confirm this is a maximum, we use the second derivative found in (ii), $\frac{d^2I}{dt^2} = Ae^{-kt}\left(k^2t - 2k\right)$, evaluated at $t = \frac{3}{2} = \frac{1}{k}$:

$$\left.\frac{d^2I}{dt^2}\right|_{t=\frac{3}{2}} = Ae^{-1}\left(k^2 \cdot \frac{1}{k} - 2k\right)$$
$$= Ae^{-1}(k - 2k)$$

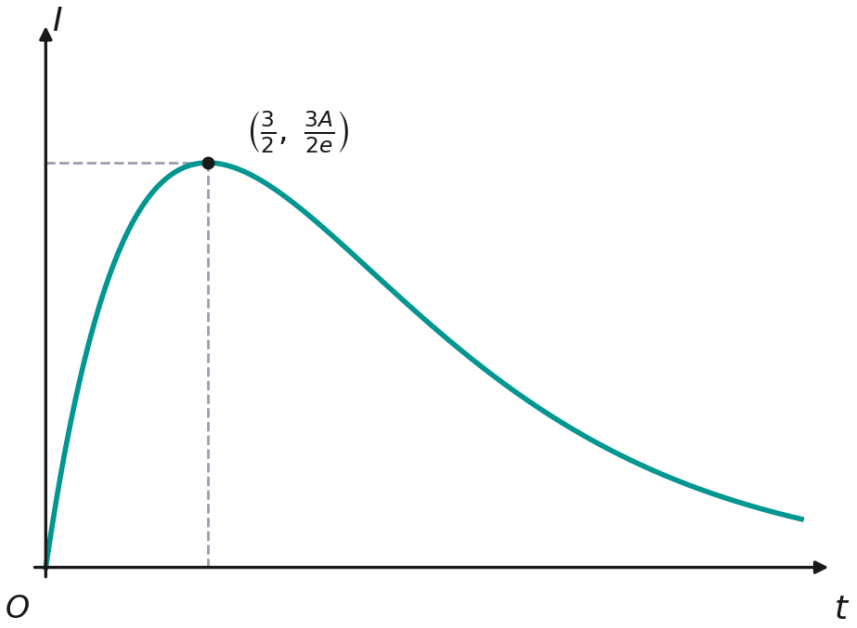
$$= -\mathrm{Ake}^{-1}$$
$$= -\frac{2A}{3e}.$$

Since $A > 0$, this is negative, so $t = \frac{3}{2}$ gives a maximum.

∴ the maximum value of I is $\frac{3A}{2e}$, attained at $t = \frac{3}{2}$.

Part (iv) 2 marks

Since t is time after the power is switched on, we sketch $I = \mathrm{Ate}^{-\frac{2}{3}t}$ for $t \geq 0$. The curve starts at the origin ($I = 0$ when $t = 0$), rises to the maximum turning point $\left(\frac{3}{2}, \frac{3A}{2e}\right)$ found in (iii), then decreases and approaches $I = 0$ as $t \to \infty$ without ever reaching it (since $e^{-\frac{2}{3}t} > 0$ for all finite t).



Question 11 16 marks

Sequences and series · savings plans

Part (i)

Let $r = 1 + \frac{a}{100}$ be the factor by which the total in the account is multiplied at the end of each month, so that with $a = 0.2$, $r = 1.002$.

Part (a)

The \$100 invested on 1 January 2016 is increased by 0.2% at the end of each of the 12 months from January to December 2016, so its value at the end of 31 December 2016 is

$$100r^{12} = 100(1.002)^{12} = \$102.43 \ \ (\text{to the nearest cent}).$$

∴ the \$100 is worth \$102.43 at the end of 31 December 2016.

Part (b)

APPROACH

The deposit made on 1st of month n (with January $=1$) sits in the account through the growth applied at the end of months $n, n+1, \ldots, 12$, so it has been multiplied by r a total of $13-n$ times by the end of December. Summing the 12 deposits gives a geometric series in r .

Since the deposit made on the 1st of month n is worth $100r^{13-n}$ at the end of December, the total at the end of 31 December 2016 is

$$\sum_{n=1}^{12} 100r^{13-n} = 100\left(r + r^2 + \cdots + r^{12}\right).$$

This is a geometric series with first term r, common ratio r and 12 terms, so

$$\text{Total} = \frac{100r\left(r^{12}-1\right)}{r-1}.$$

By GC,

$$\text{Total} = \frac{100(1.002)\left[(1.002)^{12}-1\right]}{0.002} = \$1215.71 \ \ (\text{to the nearest cent}).$$

$\therefore$ the total in the account at the end of 31 December 2016 is $\$1215.71$.

Part (c)

APPROACH

With deposits continuing every month, the total at the end of the nth month of investing is $T_n = \frac{100r(r^n-1)}{r-1}$, the same series as in (b) with 12 replaced by n. We use the GC to find the smallest n for which T_n exceeds 3000. To decide whether this happens on the first or last day of that month, we also check the amount right after the deposit is made (before that month's growth is applied) against 3000: if that alone already exceeds 3000, the crossing happens on the first day; otherwise it only happens once that month's growth is added, on the last day.

By GC, with $T_n = \frac{100r(r^n-1)}{r-1}$,

$$T_{29} = 2988.65\ldots < 3000, \qquad T_{30} = 3094.82\ldots > 3000.$$

So the total first exceeds $\$3000$ during the 30th month of investing, which (counting January 2016 as month 1) is June 2018.

To decide first day or last day, we check the amount immediately after the deposit on 1 June 2018, before that month's growth is applied. This is the balance at the end of May 2018, T_{29}, plus the new $\$100$ deposit:

$$T_{29} + 100 = 2988.65\ldots + 100 = 3088.65\ldots > 3000.$$

Note: this already exceeds $\$3000$ before June's growth is credited.

$\therefore$ the total first exceeds $\$3000$ in June 2018, and it does so on the **first day** of the month, immediately after the deposit is made (before that month's 0.2% growth is applied).

Part (ii)

Part (a)

The \$100 invested on 1 January 2016 earns a fixed bonus of $\$b$ at the end of each of the 12 months it is held in 2016 (January to December), and these bonuses do not themselves earn further bonus, so its value at the end of 31 December 2016 is

$$\$(100 + 12b).$$

(b)

Approach: The \$100 deposited on the 1st of month i (with January 2016 $=1, \ldots,$ December 2017 $=24$) is held from month i to month 24 inclusive, earning a bonus at the end of each of those $25 - i$ months. Summing over $i = 1, \ldots, 24$ gives the total bonus as b times an arithmetic series, and adding the 24 principal payments of \$100 gives the total value.

The total bonus earned by the deposit made in month i is $(25 - i)b$, so the total bonus over all 24 deposits is

$$\sum_{i=1}^{24}(25 - i)b = b\sum_{k=1}^{24} k = b \cdot \frac{24(25)}{2} = 300b,$$

using $k = 25 - i$. The total principal invested is $24(100) = 2400$, so the total value at the end of 31 December 2017 is $2400 + 300b$. Setting this equal to 2800,

$$2400 + 300b = 2800$$

$$300b = 400$$

$$b = \frac{4}{3}.$$

$\therefore b = \frac{4}{3}$, i.e. $\$1.33$ (to the nearest cent).

(iii)

Approach: With $a = 1$ for plan P, the total after 60 months of investing is the same GP sum as in (i)(b)/(i)(c) with $r = 1.01$ and $n = 60$. For plan Q, the same argument as in (ii)(b) applies with 24 replaced by 60: the total bonus is b times $\sum_{k=1}^{60} k$. Equating the two totals gives b.

For plan P with $a = 1$, $r = 1.01$, the total at the end of the 60th month is

$$T_P = \frac{100(1.01)\left[(1.01)^{60} - 1\right]}{0.01}.$$

By GC, $T_P = 8248.64$ (to the nearest cent).

For plan Q, by the same reasoning as (ii)(b) with 24 replaced by 60, the total bonus is $b\sum_{k=1}^{60} k = b \cdot \frac{60(61)}{2} = 1830b$, and the total principal is $60(100) = 6000$, so the total value at the end of the 60th month is $6000 + 1830b$. Setting the two plans equal,

$$6000 + 1830b = 8248.64\ldots$$

$$1830b = 2248.64\ldots$$

$$b = 1.2288\ldots$$

$\therefore b = \$1.23$ (to the nearest cent).

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