TEN-YEAR SERIES · WORKED SOLUTIONS

2017 A-Level H2 Mathematics Paper 1

11 questions · 100 marks

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Question 1 4 marks

Maclaurin series · standard series

Using the standard series from MF26,

$$\begin{aligned} & e^{2x} \\ &= 1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \ldots \\ &= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \ldots \end{aligned}$$

and

$$\begin{aligned} & \ln(1+ax) \\ &= (ax) - \frac{(ax)^2}{2} + \frac{(ax)^3}{3} - \ldots \\ &= ax - \frac{a^2}{2}x^2 + \frac{a^3}{3}x^3 - \ldots \end{aligned}$$

Multiplying the two series and keeping terms up to x^3, only products of powers of x adding to 3 or less are needed:

$$\begin{aligned} & e^{2x}\ln(1+ax) \\ &= \left(1 + 2x + 2x^2 + \frac{4}{3}x^3\right)\left(ax - \frac{a^2}{2}x^2 + \frac{a^3}{3}x^3\right) + \ldots \\ &= ax + \left(2a - \frac{a^2}{2}\right)x^2 + \left(2a - a^2 + \frac{a^3}{3}\right)x^3 + \ldots \end{aligned}$$

Hence, for there to be no term in x^2, the coefficient of x^2 must vanish:

$$\begin{aligned} 2a - \frac{a^2}{2} &= 0 \\ \Rightarrow a\left(2 - \frac{a}{2}\right) &= 0 \\ \Rightarrow a = 0 \ &(\text{rej. since a is non-zero}), \quad a = 4 \end{aligned}$$

$\therefore a = 4.$

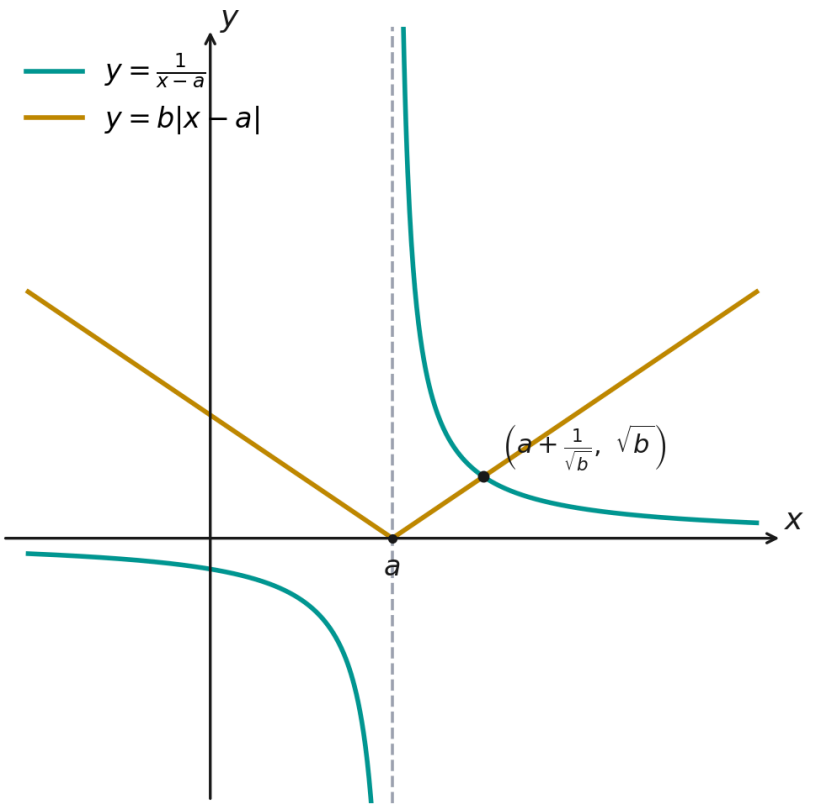
Question 2 6 marks

Inequalities · graphical solution · modulus

Part (i) 2 marks

The graph of $y = \frac{1}{x-a}$ is a rectangular hyperbola with vertical asymptote $x = a$ and horizontal asymptote $y = 0$: it lies above the x-axis for $x > a$ (decreasing towards $y = 0$ as $x \to \infty$, and $y \to \infty$ as $x \to a^+$), and below the x-axis for $x < a$ (increasing towards $y = 0$ as $x \to -\infty$, and $y \to -\infty$ as $x \to a^-$).

The graph of $y = b|x - a|$ is a V-shape with vertex $(a,\ 0)$, gradient $-b$ for $x < a$ and gradient b for $x > a$.



Part (ii) 4 marks

APPROACH

For $x < a$, $\frac{1}{x-a}$ is negative while $b|x - a|$ is non-negative, so the inequality holds for every $x < a$ with no further work. For $x > a$, both expressions are positive; the sketch in (i) shows $\frac{1}{x-a}$ starting above $b|x - a|$ near the asymptote and ending below it as $x \to \infty$, so the two graphs cross exactly once, and the inequality holds beyond that crossing. We find the crossing point algebraically.

For $x > a$, $|x - a| = x - a$, so we solve the equation

$$\frac{1}{x-a} = b(x-a)$$
$$\Rightarrow 1 = b(x-a)^2$$
$$\Rightarrow (x-a)^2 = \frac{1}{b}$$
$$\Rightarrow x - a = \frac{1}{\sqrt{b}} \ (\text{since x} > a,\ \text{so x} - a > 0)$$
$$\Rightarrow x = a + \frac{1}{\sqrt{b}}$$

Near the asymptote ($x \to a^+$), $\frac{1}{x-a} \to \infty$ while $b(x-a) \to 0$, so the curve lies above the line there and the inequality fails; beyond the crossing point the line lies above the curve, so the inequality holds for $x > a + \frac{1}{\sqrt{b}}$.

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Combining both regions:

$$x < a \text{ or } x > a + \frac{1}{\sqrt{b}}$$

To check your answer: at $x = a - 1$ (in the region $x < a$), $\frac{1}{x-a} = -1$ while $b|x - a| = b > 0$, so the inequality holds. ✓ At $x = a + \frac{2}{\sqrt{b}}$ (beyond the crossing), $\frac{1}{x-a} = \frac{\sqrt{b}}{2}$ while $b|x - a| = 2\sqrt{b}$, and $\frac{\sqrt{b}}{2} < 2\sqrt{b}$ since $b > 0$. ✓

∴ the solution is $x < a$ or $x > a + \frac{1}{\sqrt{b}}$.

Question 3 7 marks

Differentiation · implicit · stationary points

Part (i) 4 marks

APPROACH

The equation of C is not given as y in terms of x, so we differentiate implicitly to obtain $\frac{dy}{dx}$, then set it to zero.

Differentiating $y^2 - 2xy + 5x^2 - 10 = 0$ implicitly with respect to x:

$$2y\frac{dy}{dx} - 2y - 2x\frac{dy}{dx} + 10x = 0$$
$$\Rightarrow \frac{dy}{dx}(2y - 2x) = 2y - 10x$$
$$\Rightarrow \frac{dy}{dx} = \frac{2y - 10x}{2y - 2x} = \frac{y - 5x}{y - x} \quad (y \neq x)$$

At a stationary point, $\frac{dy}{dx} = 0$, so

$$y - 5x = 0$$
$$\Rightarrow y = 5x$$

Substituting into the equation of C:

$$(5x)^2 - 2x(5x) + 5x^2 - 10 = 0$$
$$\Rightarrow 25x^2 - 10x^2 + 5x^2 - 10 = 0$$
$$\Rightarrow 20x^2 = 10$$
$$\Rightarrow x^2 = \frac{1}{2}$$
$$\Rightarrow x = \pm\frac{\sqrt{2}}{2}$$

Note: at both points $y = 5x \neq x$ since $x \neq 0$, so $\frac{dy}{dx}$ is genuinely defined and equal to 0 there, confirming both are valid stationary points.

∴ $x = \frac{\sqrt{2}}{2}$ or $x = -\frac{\sqrt{2}}{2}$.

Part (ii) 3 marks

> **APPROACH**
> We differentiate $\frac{dy}{dx} = \frac{y-5x}{y-x}$ implicitly a second time to get $\frac{d^2y}{dx^2}$. Since the numerator $y - 5x$ is zero exactly at the stationary point, the term involving it drops out, leaving a short expression.

Differentiating $\frac{dy}{dx}(y - x) = y - 5x$ implicitly with respect to x:

$$\frac{d^2y}{dx^2}(y-x) + \frac{dy}{dx}\left(\frac{dy}{dx} - 1\right) = \frac{dy}{dx} - 5$$

At the stationary point, $\frac{dy}{dx} = 0$, so this reduces to

$$\frac{d^2y}{dx^2}(y-x) = -5$$
$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-5}{y-x}$$

For the stationary point with $x > 0$: $x = \frac{\sqrt{2}}{2}$, $y = 5x = \frac{5\sqrt{2}}{2}$, so

$$y - x = 4x = 2\sqrt{2} > 0$$

giving

$$\frac{d^2y}{dx^2} = \frac{-5}{2\sqrt{2}} < 0$$

Since $\frac{d^2y}{dx^2} < 0$, the stationary point with $x > 0$ is a maximum.

$\therefore$ the stationary point with $x > 0$ is a maximum point.

Question 4 8 marks

Differentiation · gradient · transformations

Part (i) 3 marks

By the quotient rule, with $u = 4x + 9$, $v = x + 2$:

$$\frac{dy}{dx}$$
$$= \frac{4(x+2) - (4x+9)(1)}{(x+2)^2}$$
$$= \frac{4x + 8 - 4x - 9}{(x+2)^2}$$
$$= \frac{-1}{(x+2)^2}$$

Since $x \neq -2$ (excluded from the domain of C), $(x + 2)^2 > 0$ for every point on C, so

$$\frac{dy}{dx} = \frac{-1}{(x+2)^2} < 0 \text{ for all points on } C.$$

∴ the gradient of C is negative for all points on C. (shown)

Part (ii) 3 marks

Writing $4x + 9 = a(x + 2) + b$:

$$4x + 9 = ax + 2a + b$$

Comparing coefficients of x: $a = 4$. Comparing constants: $2a + b = 9 \Rightarrow b = 9 - 8 = 1$.

$$\therefore y = 4 + \frac{1}{x + 2}$$

To check your answer: at $x = 0$, $y = 4 + \frac{1}{2} = \frac{9}{2}$, which matches $\frac{4(0)+9}{0+2} = \frac{9}{2}$. ✓

As $x \to \infty$, $\frac{1}{x+2} \to 0$, so $y \to 4$; as $x \to -2$, $\frac{1}{x+2} \to \pm\infty$.

∴ the asymptotes of C are $y = 4$ and $x = -2$.

Part (iii) 2 marks

APPROACH

From (ii), $y = 4 + \frac{1}{x+2}$. Starting from $y = \frac{1}{x}$, replacing x by $x + 2$ shifts the graph horizontally, and adding 4 shifts it vertically; these two translations are independent of each other and can be applied in either order.

∴ a translation of 2 units in the negative x-direction, followed by a translation of 4 units in the positive y-direction, transforms the graph of $y = \frac{1}{x}$ onto the graph of C.

Question 5 10 marks

Equations · remainder theorem · curve sketching

Part (i) 4 marks

Let $f(x) = x^3 + ax^2 + bx + c$. By the remainder theorem, dividing by $(x - 1)$, $(x - 2)$ and $(x - 3)$ gives remainders $f(1)$, $f(2)$ and $f(3)$ respectively. Using the given values:

$$f(1) = 1 + a + b + c = 8$$
$$f(2) = 8 + 4a + 2b + c = 12$$
$$f(3) = 27 + 9a + 3b + c = 25$$

Simplifying each:

$$a + b + c = 7$$
$$4a + 2b + c = 4$$
$$9a + 3b + c = -2$$

By GC (solving the system), $a = -\frac{3}{2}$, $b = \frac{3}{2}$, $c = 7$.

To check your answer: $f(1) = 1 - \frac{3}{2} + \frac{3}{2} + 7 = 8$, which matches. ✓

$\therefore a = -\frac{3}{2}$, $b = \frac{3}{2}$, $c = 7$.

Part (ii) 3 marks

Since $f(x) = x^3 - \frac{3}{2}x^2 + \frac{3}{2}x + 7$,

$$\frac{dy}{dx} = 3x^2 - 3x + \frac{3}{2}$$

Completing the square:

$$\begin{aligned} & 3x^2 - 3x + \frac{3}{2} \\ &= 3\left(x^2 - x\right) + \frac{3}{2} \\ &= 3\left(x - \frac{1}{2}\right)^2 - \frac{3}{4} + \frac{3}{2} \\ &= 3\left(x - \frac{1}{2}\right)^2 + \frac{3}{4} \end{aligned}$$

Since $\left(x - \frac{1}{2}\right)^2 \geq 0$ for all real x,

$$\frac{dy}{dx} = 3\left(x - \frac{1}{2}\right)^2 + \frac{3}{4} \geq \frac{3}{4} > 0 \text{ for all } x.$$

$\therefore$ the gradient of the curve is always positive. (shown)

Explanation: Since the gradient is always positive, the curve is strictly increasing for all x, so it can cross the x-axis at most once. A cubic with real coefficients has at least one real root, so $f(x) = 0$ has exactly one real root.

> **APPROACH**
> The root is not a nice rational number (there is no factor of $f(x)$ among ± 1, ± 7 etc., since $f(1) = 8 \neq 0$ and $f(-1) = -1.5 - 1.5 - 1.5 + 7 \neq 0$), so we solve $f(x) = 0$ numerically.

By GC, solving $x^3 - \frac{3}{2}x^2 + \frac{3}{2}x + 7 = 0$ gives $x = -1.33$ (3 s.f.).

$\therefore$ the equation $f(x) = 0$ has only one real root, $x = -1.33$ (3 s.f.).

Part (iii) 3 marks

> **APPROACH**
> The tangent is parallel to $y = 2x - 3$ when the gradient of the curve equals the gradient of the line, 2.

$$\frac{dy}{dx} = 2$$
$$3\left(x - \frac{1}{2}\right)^2 + \frac{3}{4} = 2$$
$$\left(x - \frac{1}{2}\right)^2 = \frac{5}{12}$$
$$x - \frac{1}{2} = \pm\sqrt{\frac{5}{12}}$$
$$x = \frac{1}{2} \pm \frac{\sqrt{15}}{6}$$

$\therefore$ the tangent is parallel to $y = 2x - 3$ at $x = \frac{1}{2} + \frac{\sqrt{15}}{6}$ and $x = \frac{1}{2} - \frac{\sqrt{15}}{6}$.

Question 6 8 marks

Vectors · geometrical interpretation of equations

Part (i) 2 marks

$\mathbf{r} = \mathbf{a} + t\mathbf{b}$ is the vector equation of a straight line: it passes through the point with position vector $\mathbf{a}$, and is parallel to (has direction) $\mathbf{b}$. As the parameter t varies over all real numbers, $\mathbf{r}$ traces out every point on this line.

Part (ii) 3 marks

$\mathbf{r} \cdot \mathbf{n} = d$ is the vector equation of a plane with unit normal vector $\mathbf{n}$: it consists of all points whose position vector $\mathbf{r}$ has the same projection onto $\mathbf{n}$, namely d.

Since $\mathbf{n}$ is a unit vector, $d = \mathbf{r} \cdot \mathbf{n}$ is the length of the projection of $\mathbf{r}$ onto $\mathbf{n}$ for any point on the plane, i.e. d is the perpendicular distance from the origin to the plane.

Part (iii) 3 marks

> **APPROACH**
> The point we want lies on both the line and the plane, so we substitute the line's equation for $\mathbf{r}$ into the plane's equation and solve for t.

Substituting $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ into $\mathbf{r} \cdot \mathbf{n} = d$:

$$(\mathbf{a} + t\mathbf{b}) \cdot \mathbf{n} = d$$
$$\mathbf{a} \cdot \mathbf{n} + t\,(\mathbf{b} \cdot \mathbf{n}) = d$$

Since $\mathbf{b} \cdot \mathbf{n} \neq 0$, we may divide by it:

$$t = \frac{d - \mathbf{a} \cdot \mathbf{n}}{\mathbf{b} \cdot \mathbf{n}}$$

Substituting back into $\mathbf{r} = \mathbf{a} + t\mathbf{b}$:

$$\mathbf{r} = \mathbf{a} + \left(\frac{d - \mathbf{a}\cdot\mathbf{n}}{\mathbf{b}\cdot\mathbf{n}}\right)\mathbf{b}$$

Since $\mathbf{b}\cdot\mathbf{n} \neq 0$, the line is not parallel to the plane, so it meets the plane at exactly one point.

$\therefore \mathbf{r} = \mathbf{a} + \left(\frac{d-\mathbf{a}\cdot\mathbf{n}}{\mathbf{b}\cdot\mathbf{n}}\right)\mathbf{b}$, the position vector of that unique point of intersection.

Question 7 8 marks

Integration · products of sines

Part (i) 3 marks

APPROACH

A product of two sines has no elementary antiderivative on its own; we first rewrite it as a sum using the factor formula $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$, which is easy to integrate term by term.

Using the factor formula with $A = 2mx$, $B = 2nx$:

$$\sin 2mx \sin 2nx = \frac{1}{2}\left[\cos\left(2(m-n)x\right) - \cos\left(2(m+n)x\right)\right]$$

Since $m \neq n$, both $m-n$ and $m+n$ are non-zero, so we may integrate term by term:

$$\begin{aligned}&\int \sin 2mx \sin 2nx \, dx \\ &= \frac{1}{2}\int \left[\cos\left(2(m-n)x\right) - \cos\left(2(m+n)x\right)\right] dx \\ &= \frac{\sin\left(2(m-n)x\right)}{4(m-n)} - \frac{\sin\left(2(m+n)x\right)}{4(m+n)} + C\end{aligned}$$

$\therefore \int \sin 2mx \sin 2nx \, dx = \frac{\sin(2(m-n)x)}{4(m-n)} - \frac{\sin(2(m+n)x)}{4(m+n)} + C$

Part (ii) 5 marks

APPROACH

Expanding $(f(x))^2$ gives two squared-sine terms and a cross term proportional to $\sin 2mx \sin 2nx$, which part (i) already handles. Each squared-sine term is dealt with by the double-angle identity $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$.

$$(f(x))^2 = \sin^2 2mx + 2\sin 2mx \sin 2nx + \sin^2 2nx$$

So

$$\int_0^{\pi} (f(x))^2 \, dx = \int_0^{\pi} \sin^2 2mx \, dx + 2\int_0^{\pi} \sin 2mx \sin 2nx \, dx + \int_0^{\pi} \sin^2 2nx \, dx$$

We first find $\int_0^{\pi} \sin^2 2mx \, dx$. Using $\sin^2 2mx = \frac{1}{2}(1 - \cos 4mx)$:

$$\int_0^{\pi} \sin^2 2mx\, dx$$
$$= \frac{1}{2}\left[x - \frac{\sin 4mx}{4m}\right]_0^{\pi}$$
$$= \frac{1}{2}\left(\pi - \frac{\sin 4m\pi}{4m}\right)$$

Since m is a positive integer, $\sin 4m\pi = 0$, so

$$\int_0^{\pi} \sin^2 2mx\, dx = \frac{\pi}{2}$$

By the same working with n in place of m (and $\sin 4n\pi = 0$ since n is a positive integer),

$$\int_0^{\pi} \sin^2 2nx\, dx = \frac{\pi}{2}$$

Now the cross term, using the result of (i):

$$\int_0^{\pi} \sin 2mx \sin 2nx\, dx$$
$$= \left[\frac{\sin(2(m-n)x)}{4(m-n)} - \frac{\sin(2(m+n)x)}{4(m+n)}\right]_0^{\pi}$$

Since $m-n$ and $m+n$ are non-zero integers, $\sin(2(m-n)\pi) = 0$ and $\sin(2(m+n)\pi) = 0$, and both terms vanish at $x = 0$ too, so

$$\int_0^{\pi} \sin 2mx \sin 2nx\, dx = 0$$

Combining the three results:

$$\int_0^{\pi} (f(x))^2\, dx = \frac{\pi}{2} + 2(0) + \frac{\pi}{2}$$

To check your answer: taking $m = 1$, $n = 2$ as a concrete case and evaluating $\int_0^{\pi} (\sin 2x + \sin 4x)^2\, dx$ by GC gives π, which matches. ✓

$\therefore \int_0^{\pi} (f(x))^2\, dx = \pi$

Question 8 10 marks

Complex numbers · roots · polar form

Part (a) 3 marks

APPROACH

This is a quadratic in z with complex coefficients, so we apply the quadratic formula directly (no calculator), then rationalise the denominator to reach cartesian form.

Comparing $z^2(1-i) - 2z + (5+5i) = 0$ with $Az^2 + Bz + D = 0$, $A = 1-i$, $B = -2$, $D = 5+5i$.

We first find $(1-i)(5+5i)$, needed for the discriminant:

$$
\begin{aligned}
&(1-i)(5+5i) \\
&= 5+5i-5i-5i^2 \\
&= 5+5 \\
&= 10
\end{aligned}
$$

By the quadratic formula,

$$z = \frac{2 \pm \sqrt{4-4(1-i)(5+5i)}}{2(1-i)}$$

$$B^2 - 4AD = 4 - 4(10) = -36 = 36i^2$$

so $\sqrt{B^2-4AD} = \pm 6i$, and

$$z = \frac{2 \pm 6i}{2(1-i)} = \frac{1 \pm 3i}{1-i}$$

Rationalising, multiplying numerator and denominator by $1+i$:

$$
\begin{aligned}
&z \\
&= \frac{(1 \pm 3i)(1+i)}{(1-i)(1+i)} \\
&= \frac{(1 \pm 3i)(1+i)}{2}
\end{aligned}
$$

Taking the $+$ sign: $(1+3i)(1+i) = 1+i+3i+3i^2 = -2+4i$, so $z = \frac{-2+4i}{2} = -1+2i$.

Taking the - sign: $(1-3i)(1+i) = 1+i-3i-3i^2 = 4-2i$, so $z = \frac{4-2i}{2} = 2-i$.

To check your answer: substituting $z = 2-i$: $z^2(1-i) = (3-4i)(1-i) = 3-3i-4i+4i^2 = -1-7i$, and $-2z+(5+5i) = -4+2i+5+5i = 1+7i$; summing gives $-1-7i+1+7i = 0$. ✓

$\therefore z = -1+2i$ or $z = 2-i$.

Part (b)(i) 7 marks

Since $\omega = 1-i$:

$$
\begin{aligned}
&\omega^2 \\
&= (1-i)^2 \\
&= 1-2i+i^2 \\
&= -2i \\
&\omega^3 \\
&= \omega^2 \cdot \omega \\
&= -2i(1-i) \\
&= -2i+2i^2 \\
&= -2-2i \\
&\omega^4 \\
&= \left(\omega^2\right)^2 \\
&= (-2i)^2
\end{aligned}
$$

$$= -4$$

$\therefore \omega^2 = -2i$, $\omega^3 = -2 - 2i$, $\omega^4 = -4$.

> **APPROACH**
> Substituting these into the given equation splits it into one real and one imaginary equation in p and q, since p and q are given to be real.

Substituting into $\omega^4 + p\omega^3 + 39\omega^2 + q\omega + 58 = 0$:

$$-4 + p(-2 - 2i) + 39(-2i) + q(1 - i) + 58 = 0$$
$$(54 - 2p + q) + (-2p - 78 - q)i = 0$$

Comparing real and imaginary parts (since p, q are real):

$$54 - 2p + q = 0$$
$$-2p - 78 - q = 0$$

Adding the two equations:

$$-4p - 24 = 0$$
$$p = -6$$

Substituting into $54 - 2p + q = 0$:

$$54 + 12 + q = 0$$
$$q = -66$$

To check your answer: $-2p - 78 - q = 12 - 78 - (-66) = 12 - 78 + 66 = 0$, which matches. ✓

$\therefore p = -6$, $q = -66$.

Part (b)(ii)

> **APPROACH**
> With $p = -6$ and $q = -66$, the equation $z^4 + pz^3 + 39z^2 + qz + 58 = 0$ has real coefficients and $\omega = 1 - i$ as a root, so its conjugate $\overline{\omega} = 1 + i$ is also a root. The quadratic factor with these two roots is found from their sum and product, and the other quadratic factor follows by division.

Since $z^4 - 6z^3 + 39z^2 - 66z + 58$ has real coefficients and $1 - i$ is a root, $1 + i$ is also a root, so $(z - (1 - i))(z - (1 + i))$ is a factor.

$$(z - (1 - i))(z - (1 + i))$$
$$= z^2 - [(1 - i) + (1 + i)]z + (1 - i)(1 + i)$$
$$= z^2 - 2z + 2$$

Dividing $z^4 - 6z^3 + 39z^2 - 66z + 58$ by $z^2 - 2z + 2$:

$$z^4 - 6z^3 + 39z^2 - 66z + 58 = (z^2 - 2z + 2)(z^2 + Ez + F)$$

Comparing the coefficient of z^3: $E - 2 = -6 \Rightarrow E = -4$. Comparing the constant term: $2F = 58 \Rightarrow F = 29$.

To check your answer: comparing the coefficient of z: $2E - 2F = 2(-4) - 2(29) = -8 - 58 = -66$, which matches. ✓

$\therefore z^4 - 6z^3 + 39z^2 - 66z + 58 = \left(z^2 - 2z + 2\right)\left(z^2 - 4z + 29\right).$

Question 9 13 marks

Sequences and series · sums · AP and GP

Part (a)(i) 3 marks

APPROACH

u_n is the n^{th} term of the sequence, so $u_n = S_n - S_{n-1}$ for $n \geq 2$. We then check separately whether the resulting formula also holds at $n = 1$.

For $n \geq 2$,

$$\begin{aligned} & u_n \\ &= S_n - S_{n-1} \\ &= \left(An^2 + Bn\right) - \left(A(n-1)^2 + B(n-1)\right) \\ &= An^2 + Bn - A\left(n^2 - 2n + 1\right) - Bn + B \\ &= 2An - A + B \end{aligned}$$

At $n = 1$, $u_1 = S_1 = A + B$, and the formula $2An - A + B$ at $n = 1$ gives $2A - A + B = A + B$, which agrees. So the formula holds for all $n \geq 1$.

$\therefore u_n = 2An + B - A.$

Part (a)(ii) 2 marks

Using the result of (i) with $n = 10$ and $n = 17$:

$$19A + B = 48$$
$$33A + B = 90$$

Subtracting the first equation from the second:

$$14A = 42$$
$$A = 3$$

Substituting into $19A + B = 48$:

$$19(3) + B = 48$$
$$B = -9$$

To check your answer: $33(3) + (-9) = 99 - 9 = 90$, which matches. ✓

$\therefore A = 3, B = -9.$

Part (b) 4 marks

APPROACH

The left side is a difference of two squares in $(r+1)$ and $(r-1)$, both multiplied by r^2. Factorising this way avoids expanding everything out term by term.

$$\begin{aligned} & r^2(r+1)^2 - (r-1)^2 r^2 \\ &= r^2\left[(r+1)^2 - (r-1)^2\right] \\ &= r^2\left[(r+1+r-1)(r+1-r+1)\right] \\ &= r^2(2r)(2) \\ &= 4r^3 \end{aligned}$$

$\therefore k = 4.$ (shown)

APPROACH

With $f(r) = r^2(r+1)^2$, the result just shown says $f(r) - f(r-1) = 4r^3$. Summing this from $r = 1$ to n telescopes, since every intermediate $f(r)$ cancels, leaving only the first and last terms.

Let $f(r) = r^2(r+1)^2$, so $f(r) - f(r-1) = 4r^3$.

$$\sum_{r=1}^{n} 4r^3 = \sum_{r=1}^{n} [f(r) - f(r-1)]$$

Writing out the sum,

$$\begin{aligned} & \sum_{r=1}^{n} [f(r) - f(r-1)] \\ &= [f(1) - f(0)] + [f(2) - f(1)] + \ldots + [f(n) - f(n-1)] \\ &= f(n) - f(0) \\ &= n^2(n+1)^2 - 0 \end{aligned}$$

So $4\sum_{r=1}^{n} r^3 = n^2(n+1)^2$, giving

$$\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$$

To check your answer: at $n = 3$, $\sum r^3 = 1 + 8 + 27 = 36$, and $\frac{3^2(4)^2}{4} = \frac{9 \cdot 16}{4} = 36$, which matches. ✓

$\therefore \sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}.$

Part (c) 4 marks

Here $a_r = \frac{x^r}{r!}$, so

$$\begin{aligned}&\left|\frac{a_{n+1}}{a_n}\right| \\ &= \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| \\ &= \left|\frac{x}{n+1}\right|\end{aligned}$$

For any fixed real value of x, the numerator $|x|$ is fixed while the denominator $n+1 \to \infty$ as $n \to \infty$, so

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$$

Since $0 < 1$ for every real x (including $x = 0$, where every term after a_0 is zero and the series trivially converges), the series converges for all real values of x by the ratio test.

This series is the Maclaurin series of e^x.

$\therefore \sum_{r=0}^{\infty} \frac{x^r}{r!}$ converges for all real x, with sum to infinity e^x.

Question 10 13 marks

Vectors · lines and planes in context

Part (i) 4 marks

APPROACH

Both cables are modelled as lines through known points with known directions. If they meet, there is some point lying on both lines simultaneously, so we set up a common parameter for each line and solve for where the two position vectors agree.

Let C have equation $\mathbf{r} = t\begin{pmatrix}3\\1\\-2\end{pmatrix}$, $t \in \mathbb{R}$, and let the new cable through $P(1,2,-1)$ and $Q(5,7,a)$ have direction $\overrightarrow{PQ} = \begin{pmatrix}4\\5\\a+1\end{pmatrix}$, so its equation is $\mathbf{r} = \begin{pmatrix}1\\2\\-1\end{pmatrix} + s\begin{pmatrix}4\\5\\a+1\end{pmatrix}$, $s \in \mathbb{R}$.

If the two cables meet, equating the x, y, z components gives

$$\begin{aligned}3t &= 1 + 4s \\ t &= 2 + 5s \\ -2t &= -1 + s(a+1)\end{aligned}$$

Substituting the second equation into the first:

$$3(2 + 5s) = 1 + 4s$$

$$6 + 15s = 1 + 4s$$
$$11s = -5$$
$$s = -\frac{5}{11}$$
$$t = 2 + 5\left(-\frac{5}{11}\right) = -\frac{3}{11}$$

Substituting $t = -\frac{3}{11}$ and $s = -\frac{5}{11}$ into the third equation:

$$-2\left(-\frac{3}{11}\right) = -1 - \frac{5}{11}(a+1)$$
$$\frac{6}{11} = -1 - \frac{5}{11}(a+1)$$

Multiplying throughout by 11:

$$6 = -11 - 5(a+1)$$
$$5(a+1) = -17$$
$$a + 1 = -\frac{17}{5}$$
$$a = -\frac{22}{5}$$

To check your answer: with $a = -\frac{22}{5}$, at $t = -\frac{3}{11}$ the point on C is $\left(-\frac{9}{11}, -\frac{3}{11}, \frac{6}{11}\right)$; at $s = -\frac{5}{11}$ the point on the new cable is $\left(1 - \frac{20}{11}, 2 - \frac{25}{11}, -1 - \frac{5}{11}\left(\frac{3}{5}\right)\right) = \left(-\frac{9}{11}, -\frac{3}{11}, \frac{6}{11}\right)$, which matches. ✓

$\therefore a = -\frac{22}{5}$.

Part (ii) 4 marks

APPROACH

Angle $\mathrm{PRQ} = 90^\circ$ is equivalent to $\overrightarrow{RP} \cdot \overrightarrow{RQ} = 0$. Writing R as a general point on C in terms of one parameter t, this dot product becomes a quadratic in t; showing it has no real root shows no such R exists.

With $a = -3$, $Q = (5, 7, -3)$. Let $R = t\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = (3t, t, -2t)$ be a general point on C.

$$\overrightarrow{RP} = \begin{pmatrix} 1 - 3t \\ 2 - t \\ -1 + 2t \end{pmatrix}, \quad \overrightarrow{RQ} = \begin{pmatrix} 5 - 3t \\ 7 - t \\ -3 + 2t \end{pmatrix}$$

$$\overrightarrow{RP} \cdot \overrightarrow{RQ}$$
$$= (1 - 3t)(5 - 3t) + (2 - t)(7 - t) + (-1 + 2t)(-3 + 2t)$$
$$= \left(9t^2 - 18t + 5\right) + \left(t^2 - 9t + 14\right) + \left(4t^2 - 8t + 3\right)$$
$$= 14t^2 - 35t + 22$$

For angle $\mathrm{PRQ} = 90^\circ$, we need $\overrightarrow{RP} \cdot \overrightarrow{RQ} = 0$, i.e.

$$14t^2 - 35t + 22 = 0$$

The discriminant is

$$(-35)^2 - 4(14)(22) = 1225 - 1232 = -7$$

Since the discriminant is negative, this quadratic has no real roots, so there is no real value of t, and hence no point R on C, for which $\overrightarrow{RP} \cdot \overrightarrow{RQ} = 0$.

$\therefore$ angle PRQ cannot equal 90° for any point R on C. (shown)

Part (iii) 5 marks

APPROACH

PR is smallest when R is the foot of the perpendicular from P to C. With $R = (3t, t, -2t)$, PR^2 is a quadratic in t with positive leading coefficient, so its minimum is found by differentiating and setting the derivative to zero.

With $R = (3t, t, -2t)$ as before,

$$\begin{aligned} & PR^2 \\ &= (1-3t)^2 + (2-t)^2 + (-1+2t)^2 \\ &= \left(9t^2 - 6t + 1\right) + \left(t^2 - 4t + 4\right) + \left(4t^2 - 4t + 1\right) \\ &= 14t^2 - 14t + 6 \end{aligned}$$

Differentiating with respect to t and setting to zero:

$$\frac{d}{dt}\left(PR^2\right) = 28t - 14 = 0$$

$$t = \frac{1}{2}$$

Since the coefficient of t^2 is positive, this is a minimum.

$$R = \left(3\left(\frac{1}{2}\right),\ \frac{1}{2},\ -2\left(\frac{1}{2}\right)\right) = \left(\frac{3}{2},\ \frac{1}{2},\ -1\right)$$

$$PR^2 = 14\left(\frac{1}{4}\right) - 14\left(\frac{1}{2}\right) + 6 = \frac{7}{2} - 7 + 6 = \frac{5}{2}$$

$$PR = \sqrt{\frac{5}{2}} = \frac{\sqrt{10}}{2}$$

To check your answer: $\overrightarrow{PR} = \left(\frac{3}{2} - 1,\ \frac{1}{2} - 2,\ -1 - (-1)\right) = \left(\frac{1}{2},\ -\frac{3}{2},\ 0\right)$, and this is perpendicular to the direction of C: $\left(\frac{1}{2}\right)(3) + \left(-\frac{3}{2}\right)(1) + 0(-2) = \frac{3}{2} - \frac{3}{2} = 0$, which matches. ✓

$\therefore R = \left(\frac{3}{2},\ \frac{1}{2},\ -1\right)$, and the minimum length of PR is $\frac{\sqrt{10}}{2}\ m$.

Question 11 13 marks

Differential equations · motion under gravity

Part (i)(a) 1 mark

$$\frac{dv}{dt} = c$$

Part (i)(b) 3 marks

APPROACH

With $\frac{dv}{dt} = c$ constant, integrating directly gives v as a linear function of t; the initial condition and the value after $2.5\ s$ then pin down the two constants.

$$\frac{dv}{dt} = c$$
$$\int dv = \int c\,dt$$
$$v = ct + D$$

Since $v = 4$ when $t = 0$, $D = 4$, so $v = ct + 4$.

Using the given value $v = 29$ when $t = 2.5$:

$$29 = 2.5c + 4$$
$$c = 10$$

$\therefore v = 10t + 4$, with $c = 10$.

Part (ii) 5 marks

APPROACH

The new rate of change is $\frac{dv}{dt} = 10 - kv$, a first-order equation with v and t separable. We integrate both sides, using the initial condition $v = 0$ at $t = 0$ to fix the constant, then rearrange to make v the subject.

$$\frac{dv}{dt} = 10 - kv$$

Separating variables,

$$\int \frac{1}{10 - kv}\,dv = \int dt$$
$$-\frac{1}{k}\ln|10 - kv| = t + D$$

Since $v = 0$ when $t = 0$:

$$-\frac{1}{k}\ln 10 = D$$

Substituting back:

$$-\frac{1}{k}\ln|10-kv| = t - \frac{1}{k}\ln 10$$

Multiplying throughout by $-k$:

$$\ln|10-kv| = -kt + \ln 10$$
$$\ln\left|\frac{10-kv}{10}\right| = -kt$$

Note: while the terminal velocity has not yet been reached, $v < \frac{10}{k}$, so $10 - kv > 0$ and the modulus can be dropped.

$$\frac{10-kv}{10} = e^{-kt}$$
$$10 - kv = 10e^{-kt}$$
$$v = \frac{10}{k}\left(1 - e^{-kt}\right)$$

To check your answer: at $t = 0$, $v = \frac{10}{k}(1-1) = 0$, which matches the given initial condition. ✓

$\therefore v = \frac{10}{k}\left(1 - e^{-kt}\right)$.

Part (iii) 4 marks

APPROACH

The terminal velocity is the limit of v as $t \to \infty$. As $t \to \infty$, $e^{-kt} \to 0$ (since $k > 0$), so this limit is $\frac{10}{k}$, which lets us find k from the given terminal velocity before solving for t.

As $t \to \infty$, $e^{-kt} \to 0$, so the terminal velocity is

$$\lim_{t\to\infty} v = \frac{10}{k}$$

Since the terminal velocity is $40\ m\,s^{-1}$:

$$\frac{10}{k} = 40$$
$$k = \frac{1}{4}$$

So $v = 40\left(1 - e^{-t/4}\right)$. We require t such that $v = 0.9(40) = 36$:

$$36 = 40\left(1 - e^{-t/4}\right)$$
$$1 - e^{-t/4} = 0.9$$
$$e^{-t/4} = 0.1$$
$$-\frac{t}{4} = \ln 0.1$$
$$t = -4\ln 0.1 = 4\ln 10$$

$\therefore t = 4\ln 10 = 9.21\ s$ (3 s.f.).

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