TEN-YEAR SERIES · WORKED SOLUTIONS

2016 A-Level H2 Mathematics Paper 2

10 questions · 100 marks

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Question 1 7 marks

Differentiation · rates of change

APPROACH

This is a related-rates problem. We are given $\frac{dV}{dt}$ and asked for $\frac{dh}{dt}$, so we need V as a function of h alone. The diagram shows the semi-vertical angle α is measured from the axis of the cone to the slant, so $\tan\alpha = \frac{r}{h}$; this lets us eliminate r from the volume formula before differentiating.

Since $\tan\alpha = \frac{r}{h} = 0.5$, we have $r = \frac{1}{2}h$.

Substituting into the given volume formula:

$$\begin{aligned} V &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h \\ &= \frac{1}{12}\pi h^3 \end{aligned}$$

Differentiating with respect to h:

$$\frac{dV}{dh} = \frac{1}{4}\pi h^2$$

When $V = 3$:

$$\begin{aligned} 3 &= \frac{1}{12}\pi h^3 \\ h^3 &= \frac{36}{\pi} \\ h &= \left(\frac{36}{\pi}\right)^{\frac{1}{3}} \end{aligned}$$

By GC, $h = 2.2545\ldots$

Using the chain rule, with $\frac{dV}{dt} = 0.1$:

$$\begin{aligned} \frac{dh}{dt} &= \frac{dh}{dV} \times \frac{dV}{dt} \\ &= \frac{1}{\frac{1}{4}\pi h^2} \times 0.1 \\ &= \frac{0.4}{\pi h^2} \end{aligned}$$

By GC, substituting $h = 2.2545\ldots$:

$$\frac{dh}{dt} = 0.0251 \ \ (3\text{ s.f.})$$

To check your answer: near $V = 3$, a small increase $\Delta V = 0.1$ should raise h by roughly 0.025; since $h \approx 2.25$, this is under 2% of h over a full minute, consistent with the cone widening quickly as it fills. ✓

$\therefore$ the depth of water is increasing at $0.0251\ \text{m min}^{-1}$ (3 s.f.) when $V = 3\ m^3$.

Question 2 10 marks

Integration · by parts · definite integrals

Part (a)(i) 3 marks

APPROACH

$x^2 \cos nx$ is a product of a polynomial and a trigonometric function, so we integrate by parts, choosing x^2 as the part that differentiates to something simpler and applying the process twice.

Let $u = x^2$ and $\frac{dv}{dx} = \cos nx$, so $\frac{du}{dx} = 2x$ and $v = \frac{1}{n}\sin nx$.

$$\int x^2 \cos nx\, dx$$

$$= \frac{x^2}{n}\sin nx - \int \frac{2x}{n}\sin nx\, dx$$

For $\int x \sin nx\, dx$, integrate by parts again with $u = x$, $\frac{dv}{dx} = \sin nx$, so $\frac{du}{dx} = 1$, $v = -\frac{1}{n}\cos nx$:

$$\int x \sin nx\, dx$$

$$= -\frac{x}{n}\cos nx + \int \frac{1}{n}\cos nx\, dx$$

$$= -\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx$$

Substituting back:

$$\int x^2 \cos nx\, dx$$

$$= \frac{x^2}{n}\sin nx - \frac{2}{n}\left(-\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx\right) + C$$

$$= \frac{x^2}{n}\sin nx + \frac{2x}{n^2}\cos nx - \frac{2}{n^3}\sin nx + C$$

$\therefore \int x^2 \cos nx\, dx = \frac{x^2}{n}\sin nx + \frac{2x}{n^2}\cos nx - \frac{2}{n^3}\sin nx + C$.

Part (a)(ii) 2 marks

APPROACH

We evaluate the antiderivative from (i) at $x = 2\pi$ and $x = \pi$. Since n is an integer, $\sin n\pi = \sin 2n\pi = 0$ for every n, so only the $\cos nx$ term survives; but $\cos 2n\pi = 1$ always, while $\cos n\pi = (-1)^n$ depends on the parity of n, so the two cases must be treated separately.

Using the result of (i):

$$\int_{\pi}^{2\pi} x^2 \cos nx \, dx = \left[\frac{x^2}{n}\sin nx + \frac{2x}{n^2}\cos nx - \frac{2}{n^3}\sin nx\right]_{\pi}^{2\pi}$$

Since n is an integer, $\sin 2n\pi = \sin n\pi = 0$, so this reduces to

$$\begin{aligned}\int_{\pi}^{2\pi} x^2 \cos nx \, dx &= \frac{4\pi}{n^2}\cos 2n\pi - \frac{2\pi}{n^2}\cos n\pi \\ &= \frac{4\pi}{n^2} - \frac{2\pi}{n^2}\cos n\pi\end{aligned}$$

Note: $\cos n\pi = (-1)^n$, which is 1 when n is even and -1 when n is odd.

If n is even:

$$\int_{\pi}^{2\pi} x^2 \cos nx \, dx = \frac{4\pi}{n^2} - \frac{2\pi}{n^2} = \frac{2\pi}{n^2}$$

If n is odd:

$$\int_{\pi}^{2\pi} x^2 \cos nx \, dx = \frac{4\pi}{n^2} + \frac{2\pi}{n^2} = \frac{6\pi}{n^2}$$

$\therefore \int_{\pi}^{2\pi} x^2 \cos nx \, dx = a\frac{\pi}{n^2}$, where $a = 2$ if n is even, and $a = 6$ if n is odd.

Part (b) 5 marks

APPROACH

We need $\int_0^2 \pi y^2 \, dx = \pi \int_0^2 \frac{x^3}{(9-x^2)^2} \, dx$. The given substitution $u = 9 - x^2$ turns $9 - x^2$ into u and, since $du = -2x \, dx$, turns the leftover $x^2 \cdot x \, dx$ into $(9 - u)$ times a constant times du — this is why the substitution is chosen: it clears the awkward x^2 inside the bracket as well as supplying the $x \, dx$.

Let $y = \frac{x\sqrt{x}}{9-x^2}$, so $y^2 = \frac{x^3}{(9-x^2)^2}$.

Volume $= \pi \int_0^2 y^2 \, dx = \pi \int_0^2 \frac{x^3}{(9-x^2)^2} \, dx$.

Let $u = 9 - x^2$. Then $\frac{du}{dx} = -2x$, so $x \, dx = -\frac{1}{2} du$, and $x^2 = 9 - u$.

When $x = 0$, $u = 9$; when $x = 2$, $u = 5$.

$$\begin{aligned}\int_0^2 \frac{x^3}{(9-x^2)^2} \, dx &= \int_0^2 \frac{x^2}{(9-x^2)^2} \cdot x \, dx \\ &= \int_9^5 \frac{9-u}{u^2}\left(-\frac{1}{2}\right) du \\ &= \frac{1}{2}\int_5^9 \left(\frac{9}{u^2} - \frac{1}{u}\right) du \\ &= \frac{1}{2}\left[-\frac{9}{u} - \ln u\right]_5^9\end{aligned}$$

Evaluating:

$$\frac{1}{2}\left[-\frac{9}{u} - \ln u\right]_5^9$$

$$= \frac{1}{2}\left[(-1 - \ln 9) - \left(-\frac{9}{5} - \ln 5\right)\right]$$
$$= \frac{1}{2}\left[\frac{4}{5} + \ln 5 - \ln 9\right]$$
$$= \frac{1}{2}\left(\frac{4}{5} + \ln\frac{5}{9}\right)$$
$$= \frac{2}{5} - \frac{1}{2}\ln\frac{9}{5}$$

So Volume $= \pi\left(\frac{2}{5} - \frac{1}{2}\ln\frac{9}{5}\right)$.

To check your answer: numerically this is $\pi(0.4 - 0.2939\ldots) = \pi(0.1061\ldots) \approx 0.333$, which is positive and small, consistent with a thin sliver of solid swept from a curve with $y(0) = 0$ and $y(2) = \frac{2\sqrt{2}}{5} \approx 0.566$. ✓

$\therefore$ the exact volume is $\pi\left(\frac{2}{5} - \frac{1}{2}\ln\frac{9}{5}\right)$ units^3.

Question 3 11 marks

Parametric equations · sketch · area · normal

Part (i) 4 marks

APPROACH

Before sketching we find where the curve crosses the x-axis, its maximum point, and how x behaves as t increases, since D is given parametrically and is not a graph of a simple function.

D meets the x-axis where $y = 0$:

$$y = 1 - \cos t = 0$$
$$\cos t = 1$$
$$t = 0 \text{ or } t = 2\pi \quad (0 \le t \le 2\pi)$$

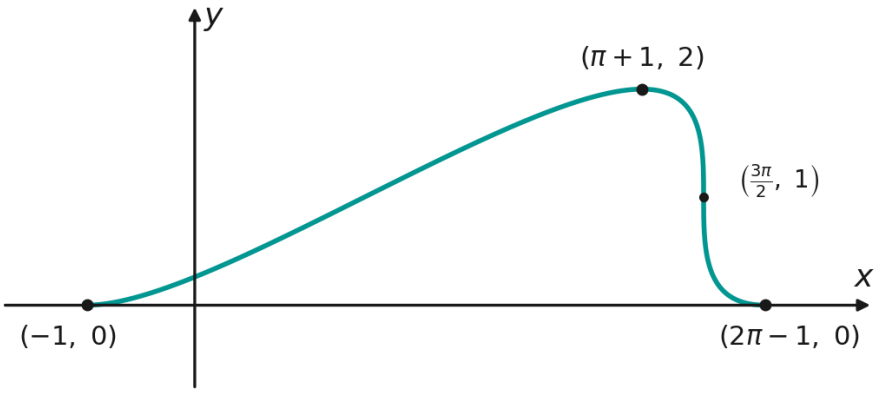
At $t = 0$: $x = 0 - \cos 0 = -1$. At $t = 2\pi$: $x = 2\pi - \cos 2\pi = 2\pi - 1$.

$\therefore$ D meets the x-axis at $(-1,\ 0)$ and $(2\pi - 1,\ 0)$.

Since $y = 1 - \cos t$ and $-1 \le \cos t \le 1$, y is greatest when $\cos t = -1$, i.e. at $t = \pi$, giving $y = 2$. At $t = \pi$: $x = \pi - \cos\pi = \pi + 1$.

$\therefore$ the maximum point is $(\pi + 1,\ 2)$.

Note: $\frac{dx}{dt} = 1 + \sin t \ge 0$ for all t, with equality only at $t = \frac{3\pi}{2}$, so x increases throughout $[0, 2\pi]$ and D does not double back on itself. At $t = \frac{3\pi}{2}$ the curve has a vertical tangent, at the point $\left(\frac{3\pi}{2},\ 1\right)$.



Part (ii) 3 marks

APPROACH

The area under a parametric curve is $\int y\,dx = \int y\,\frac{dx}{dt}\,dt$, taken over the corresponding range of t.

$$\begin{aligned}\text{Area} &= \int_0^a y\,\frac{dx}{dt}\,dt \\ &= \int_0^a (1-\cos t)\,(1+\sin t)\;dt\end{aligned}$$

Expanding the integrand:

$$(1-\cos t)\,(1+\sin t) = 1+\sin t-\cos t-\sin t\cos t$$

So

$$\begin{aligned}\text{Area} &= \int_0^a (1+\sin t-\cos t-\sin t\cos t)\,dt \\ &= \left[t-\cos t-\sin t-\frac{1}{2}\sin^2 t\right]_0^a \\ &= \left(a-\cos a-\sin a-\frac{1}{2}\sin^2 a\right)-(0-1-0-0) \\ &= a+1-\sin a-\cos a-\frac{1}{2}\sin^2 a\end{aligned}$$

$\therefore$ Area $= a+1-\sin a-\cos a-\frac{1}{2}\sin^2 a$.

Part (iii) 4 marks

APPROACH

We find the point on D at $t=\frac{\pi}{2}$ and the gradient of the tangent there using $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}$, then take the negative reciprocal for the normal. Since E and F are the x- and y-intercepts of the normal, triangle OEF is right-angled at O, so its area is simply $\frac{1}{2}\,|OE|\,|OF|$.

At $t=\frac{\pi}{2}$: $x=\frac{\pi}{2}-\cos\frac{\pi}{2}=\frac{\pi}{2}$, and $y=1-\cos\frac{\pi}{2}=1$. So the point is $\left(\frac{\pi}{2},\,1\right)$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sin t}{1+\sin t}$$

At $t = \frac{\pi}{2}$: $\frac{dy}{dx} = \frac{1}{1+1} = \frac{1}{2}$.

Since the normal is perpendicular to the tangent, its gradient is -2.

Equation of the normal:

$$y - 1 = -2\left(x - \frac{\pi}{2}\right)$$

At E (on the x-axis), $y = 0$:

$$0 - 1 = -2\left(x - \frac{\pi}{2}\right)$$
$$x = \frac{\pi}{2} + \frac{1}{2}$$

So $E = \left(\frac{\pi+1}{2},\ 0\right)$.

At F (on the y-axis), $x = 0$:

$$y - 1 = -2\left(0 - \frac{\pi}{2}\right)$$
$$y = 1 + \pi$$

So $F = (0,\ 1+\pi)$.

Since OEF is right-angled at O:

$$\text{Area of OEF} = \frac{1}{2}\,|OE|\,|OF|$$
$$= \frac{1}{2} \cdot \frac{\pi+1}{2} \cdot (\pi+1)$$
$$= \frac{(\pi+1)^2}{4}$$

To check your answer: numerically $\frac{(\pi+1)^2}{4} \approx 4.29$, and directly multiplying the intercepts $\frac{1}{2}(2.071)(4.142) \approx 4.29$. ✓

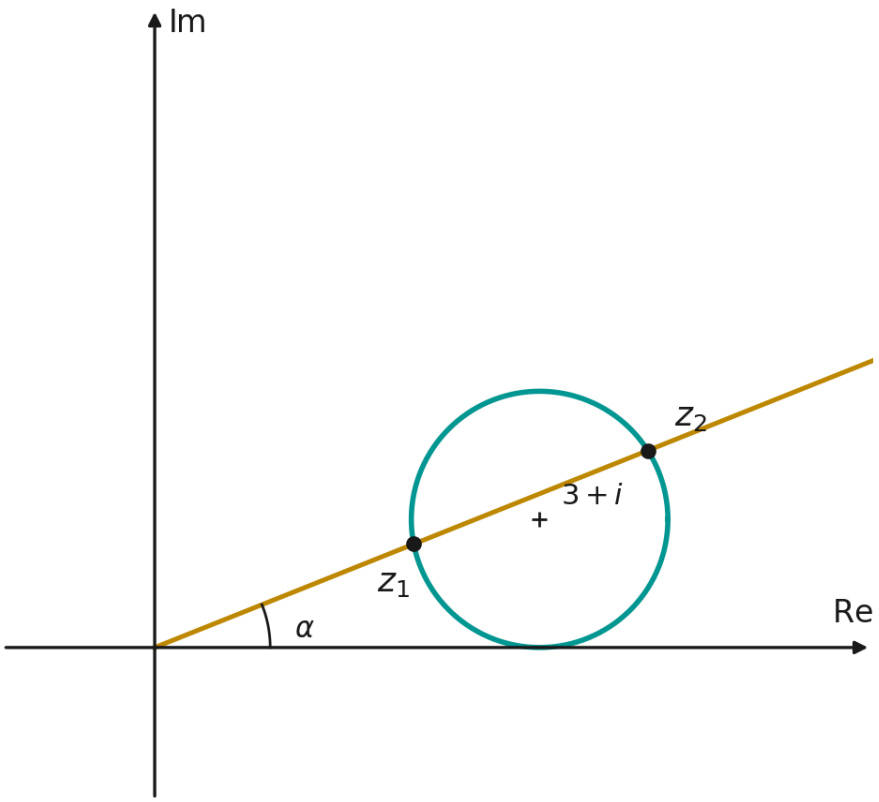
$\therefore$ the exact area of triangle OEF is $\frac{(\pi+1)^2}{4}$ units^2.

Question 4 12 marks

Complex numbers · loci in the Argand diagram

Part (a)(i) 2 marks

The locus $|z - 3 - i| = 1$ is a circle centre $3 + i$, radius 1. The locus $\arg z = \alpha$, where $\tan\alpha = 0.4$, is a half-line (ray) from the origin, in the first quadrant since α is acute, at angle $\alpha \approx 21.8°$ to the positive real axis. Since $|z_1| < |z_2|$, z_1 is the nearer intersection point to the origin.



Part (a)(ii) 4 marks

APPROACH

z_1 and z_2 both lie on the circle $|z-3-i|=1$, so any point equidistant from z_1 and z_2 lies on the perpendicular bisector of the chord z_1z_2 — and that perpendicular bisector passes through the centre $3+i$, since the centre is itself equidistant from every point on the circle. The chord z_1z_2 lies along the ray at angle α, so the perpendicular bisector through the centre has direction $\alpha+\frac{\pi}{2}$; the required points are where this line, extended both ways from the centre, meets the circle again — i.e. the centre displaced by ± 1 (the radius) in the direction $\alpha+\frac{\pi}{2}$.

Since $\tan\alpha = 0.4 = \frac{2}{5}$, take a right triangle with opposite 2, adjacent 5, hypotenuse $\sqrt{4+25} = \sqrt{29}$:

$$\sin\alpha = \frac{2}{\sqrt{29}}, \qquad \cos\alpha = \frac{5}{\sqrt{29}}$$

A unit vector in the direction $\alpha+\frac{\pi}{2}$ is $(-\sin\alpha,\ \cos\alpha)$, since rotating by $\frac{\pi}{2}$ sends $(\cos\alpha, \sin\alpha) \mapsto (-\sin\alpha, \cos\alpha)$.

The required points of $|z-3-i|=1$ are the centre $3+i$ displaced by ± 1 in this direction:

$$\begin{aligned} z &= (3+i) \pm i\,(\cos\alpha + i\sin\alpha) \\ &= 3+i \pm (-\sin\alpha + i\cos\alpha) \\ &= \left(3 \mp \frac{2}{\sqrt{29}}\right) + i\left(1 \pm \frac{5}{\sqrt{29}}\right) \end{aligned}$$

$\therefore$ the two values of z are

$$z = \left(3 - \frac{2}{\sqrt{29}}\right) + i\left(1 + \frac{5}{\sqrt{29}}\right) \quad \text{and} \quad z = \left(3 + \frac{2}{\sqrt{29}}\right) + i\left(1 - \frac{5}{\sqrt{29}}\right).$$

To check your answer: both points satisfy $|z - 3 - i| = |\pm(-\sin\alpha + i\cos\alpha)| = \sqrt{\sin^2\alpha + \cos^2\alpha} = 1$, confirming they lie on the given circle. ✓

Part (b)(i) 3 marks

APPROACH

To find cube roots we write w in polar form first, then use $w^{1/3} = r^{1/3}e^{i(\theta+2k\pi)/3}$ for $k = 0, \pm 1$, choosing k so that each resulting argument lands in $(-\pi, \pi]$.

Since $w = 2 - 2i$:

$$r = |w| = \sqrt{2^2 + 2^2} = 2\sqrt{2}$$

w lies in the fourth quadrant, with $\tan\theta = \frac{-2}{2} = -1$, so $\theta = -\frac{\pi}{4}$.

$$\therefore w = 2\sqrt{2}\,e^{-i\frac{\pi}{4}}$$

The cube roots of w have modulus $r^{1/3} = \left(2\sqrt{2}\right)^{1/3} = \left(2^{3/2}\right)^{1/3} = 2^{1/2} = \sqrt{2}$, and arguments $\frac{\theta+2k\pi}{3}$ for $k = 0, \pm 1$:

$$k = 0: \quad \frac{-\pi/4}{3} = -\frac{\pi}{12}$$

$$k = 1: \quad \frac{-\pi/4 + 2\pi}{3} = \frac{7\pi}{12}$$

$$k = -1: \quad \frac{-\pi/4 - 2\pi}{3} = -\frac{3\pi}{4}$$

All three arguments lie in $(-\pi, \pi]$.

$\therefore$ the cube roots of w are

$$\sqrt{2}\,e^{-i\frac{\pi}{12}}, \qquad \sqrt{2}\,e^{i\frac{7\pi}{12}}, \qquad \sqrt{2}\,e^{-i\frac{3\pi}{4}}.$$

Part (b)(ii) 3 marks

APPROACH

We write w^*w^n in polar form so its argument is a linear expression in n, then find the smallest positive integer n for which that argument reduces to $\frac{\pi}{2}$ inside the principal range, remembering to add or subtract multiples of 2π as needed.

Since $w = 2\sqrt{2}\,e^{-i\pi/4}$, its conjugate is $w^* = 2\sqrt{2}\,e^{i\pi/4}$.

$$w^*w^n = 2\sqrt{2}\,e^{i\pi/4} \cdot \left(2\sqrt{2}\right)^n e^{-in\pi/4}$$

$$= \left(2\sqrt{2}\right)^{n+1} e^{i\frac{\pi}{4}(1-n)}$$

We require $\arg\left(w^*w^n\right) = \frac{\pi}{2}$, i.e.

$$\frac{\pi}{4}(1-n) = \frac{\pi}{2} + 2k\pi, \quad k \in \mathbb{Z}$$

$$1 - n = 2 + 8k$$

$$n = -1 - 8k$$

For n to be a smallest **positive** whole number, take $k = -1$:

$$n = -1 - 8(-1) = 7$$

To check your answer: at $n = 7$, $\frac{\pi}{4}(1-7) = -\frac{3\pi}{2}$, and $-\frac{3\pi}{2} + 2\pi = \frac{\pi}{2}$, the required argument. ✓

$\therefore$ the smallest positive whole number value of n is 7.

Question 5 5 marks

Discrete random variables · expectation and variance

Let R, B, Y be the events that the spinner comes to rest over the sections labelled $\mathbf{R}$, $\mathbf{B}$, $\mathbf{Y}$ respectively, and let W be the event that the player wins the game.

Since the spinner has 7 equally likely sections, 1 labelled $\mathbf{R}$, 2 labelled $\mathbf{B}$ and 4 labelled $\mathbf{Y}$,

$$P(R) = \frac{1}{7}, \qquad P(B) = \frac{2}{7}, \qquad P(Y) = \frac{4}{7}.$$

Since the red die has 3 faces with *, the blue die has 2 faces with *, and the yellow die has 1 face with *, each out of 6 faces,

$$P(W \mid R) = \frac{3}{6}, \qquad P(W \mid B) = \frac{2}{6}, \qquad P(W \mid Y) = \frac{1}{6}.$$

Part (i) 2 marks

By the law of total probability, since R, B, Y partition the sample space:

$$\begin{aligned} & P(W) \\ &= P(R)P(W \mid R) + P(B)P(W \mid B) + P(Y)P(W \mid Y) \\ &= \frac{1}{7} \times \frac{3}{6} + \frac{2}{7} \times \frac{2}{6} + \frac{4}{7} \times \frac{1}{6} \\ &= \frac{11}{42} \end{aligned}$$

$\therefore\ P(W) = \frac{11}{42}$.

Part (ii) 1 mark

Using Bayes' theorem,

$$\begin{aligned} & P(B \mid W) \\ &= \frac{P(B)P(W \mid B)}{P(W)} \\ &= \frac{\frac{2}{7} \times \frac{2}{6}}{\frac{11}{42}} \\ &= \frac{4}{11} \end{aligned}$$

$\therefore\ P(B \mid W) = \frac{4}{11}$.

Part (iii) 2 marks

APPROACH

"A different colour die each time" over the 3 games means the spinner shows $\mathbf{R}$, $\mathbf{B}$, $\mathbf{Y}$ in some order, one each, over the 3 games — there are $3! = 6$ such orders, and by independence of the games each order has the same probability. So rather than listing 6 cases, we find the probability of one representative order and multiply by $3!$.

Let $p_R = P(R)P(W \mid R)$, $p_B = P(B)P(W \mid B)$, $p_Y = P(Y)P(W \mid Y)$, the probability that a single game is won with the red, blue, yellow die respectively.

$$p_R = \frac{1}{7} \times \frac{3}{6} = \frac{1}{14}, \qquad p_B = \frac{2}{7} \times \frac{2}{6} = \frac{2}{21}, \qquad p_Y = \frac{4}{7} \times \frac{1}{6} = \frac{2}{21}$$

Since the 3 games are independent, and there are $3!$ ways to assign the colours $\mathbf{R}$, $\mathbf{B}$, $\mathbf{Y}$ to the 3 games, all winning games:

$$\begin{aligned} &P\left(\text{win 3 games, each a different colour die}\right) \\ &= 3! \times p_R \times p_B \times p_Y \\ &= 6 \times \frac{1}{14} \times \frac{2}{21} \times \frac{2}{21} \\ &= \frac{4}{1029} \end{aligned}$$

To check your answer: $\frac{4}{1029} \approx 0.00389$, sensibly small since it requires winning all 3 games with 3 different, specific colours. ✓

$\therefore$ the required probability is $\frac{4}{1029}$.

Question 6 10 marks

Probability · conditional probability from a table

Part (i)(a)

Since the sample is stratified by department and gender, the number of males in the sample is found in proportion to the total number of males among all $2345 + 1013 + 237 + 344 + 867 + 679 + 591 + 523 = 6599$ employees.

$$\text{no. of males} = \frac{2345 + 1013 + 237 + 344}{6599} \times 100 = \frac{3939}{6599} \times 100 = 59.7 \ \ (3\text{ s.f.})$$

Since the number sampled must be a whole number, this rounds to 60.

$\therefore$ the sample should contain 60 males.

Part (i)(b)

$$\text{no. of females from Development} = \frac{679}{6599} \times 100 = 10.3 \ \ (3 \text{ s.f.})$$

Since the number sampled must be a whole number, this rounds to 10.

> $\therefore$ the sample should contain 10 females from the Development department.

Part (ii)

The stratified sample in **(i)** should not be used, since it is stratified by department and gender, not by age.

Explanation: Stratifying by department and gender gives no assurance about how ages are distributed within each stratum, so the sample in **(i)** is not designed to be representative of the age distribution of the company's employees, and is not suitable for testing a belief about the mean age.

Part (iii)

Let μ be the population mean age, in years, of all employees of the company now, and let $\overline{X}$ be the mean age of a random sample of 80 employees.

Since the MD believes the mean age now is less than 37, we test

$$H_0 : \mu = 37 \qquad \text{against} \qquad H_1 : \mu < 37$$

at the 5% significance level.

> **APPROACH**
>
> The population variance is given, so we use a Z-test. Since $n = 80$ is large, by the Central Limit Theorem $\overline{X}$ is approximately normally distributed even though the population distribution of ages is not given, and the test statistic below is approximately standard normal under H_0.

Since $n = 80$ is large, by the Central Limit Theorem, under H_0,

$$Z = \frac{\overline{X} - 37}{\sqrt{140/80}} \sim N(0, 1) \text{ approximately.}$$

Since H_1 is one-tailed (lower), we reject H_0 at the 5% level if $z \leq -1.6449$ (by GC, $z_{0.05} = 1.6449$), i.e. if

$$\frac{\overline{x} - 37}{\sqrt{140/80}} \leq -1.6449$$

$$\overline{x} \leq 37 - 1.6449\sqrt{\frac{140}{80}}$$

$$\leq 34.8 \ \ (3 \text{ s.f.})$$

We are given that the test result is that the MD's belief should be accepted, i.e. H_0 is rejected.

> $\therefore$ the set of possible values of $\overline{x}$ is $\{\overline{x} \in \mathbb{R} : \overline{x} \leq 34.8\}$.

Part (iv)

APPROACH

Here the sample mean $\overline{x} = 35.2$ is fixed and we are told H_0 is *not* rejected at the $\alpha\%$ level; we need the range of α for which this happens, which we find by comparing the (fixed) p-value of this sample to $\alpha/100$.

With $\overline{x} = 35.2$,

$$z = \frac{35.2 - 37}{\sqrt{140/80}} = -1.3607 \ (5 \text{ s.f.})$$

By GC, the p-value is

$$p = P(Z \leq -1.3607) = 0.0868 \ (3 \text{ s.f.})$$

H_0 is rejected (MD's belief accepted) if $p < \frac{\alpha}{100}$, so the MD's belief is **not** accepted if $p \geq \frac{\alpha}{100}$, i.e.

$$\alpha \leq 100 \times 0.0868$$

$$\alpha \leq 8.68 \ (3 \text{ s.f.})$$

Since α is a significance level expressed as a percentage, $\alpha > 0$.

$\therefore$ the set of possible values of α is $\{\alpha \in \mathbb{R} : 0 < \alpha \leq 8.68\}$.

Question 7 10 marks

Permutations and combinations · selections and arrangements

The board has 6 men and 4 women, 10 members in total. A chairperson, a secretary and a treasurer are three different roles, so choosing who fills them is an arrangement (order matters), not just a selection.

Part (i) 1 mark

Since all three roles are filled by women, we are arranging 3 of the 4 women into 3 distinct roles: 4 choices for chairperson, then 3 for secretary, then 2 for treasurer.

$$\text{no. of ways} = 4 \times 3 \times 2 = 24$$

$\therefore$ there are 24 ways.

Part (ii) 3 marks

APPROACH

"At least one is a woman and at least one is a man" is easiest by the complement: from all ways of filling the 3 roles, remove the ways that are all men and the ways that are all women (these two cases are mutually exclusive and together are the complement of "at least one of each").

The total number of ways to fill the 3 roles from all 10 members is

$$10 \times 9 \times 8 = 720$$

The number of ways with all 3 roles filled by men is

$$6 \times 5 \times 4 = 120$$

and from part **(i)**, the number of ways with all 3 roles filled by women is 24.

$$\text{no. of ways} = 720 - 120 - 24 = 576$$

$\therefore$ there are 576 ways.

Part (iii) 3 marks

APPROACH

The 10 members sit at random around a (circular) table, so we count circular arrangements, where only rotations of the same seating are identified (reflections are distinct, since swapping left and right neighbours gives a genuinely different seating). The total number of such arrangements of 10 distinct people is $(10-1)! = 9!$. For the favourable outcomes, we treat the chairperson, secretary and treasurer as a single block, since they must sit in three adjacent places.

Treating the chairperson, secretary and treasurer as one block leaves 8 objects (the block and the other 7 members) to arrange around the table:

$$\text{no. of arrangements of the 8 objects} = (8-1)! = 7!$$

Within the block, the 3 officers can be ordered in $3!$ ways.

$$\begin{aligned} P(\text{3 officers adjacent}) &= \frac{7! \times 3!}{9!} \\ &= \frac{7! \times 6}{9 \times 8 \times 7!} \\ &= \frac{6}{72} \\ &= \frac{1}{12} \end{aligned}$$

$\therefore$ $P(\text{chairperson, secretary and treasurer sit in three adjacent places}) = \frac{1}{12}$.

Part (iv) 3 marks

APPROACH

"Separated from each other by at least one other person" means no two of the three officers are adjacent. Rather than working with the officers directly, we first seat the other 7 members around the table, which creates 7 gaps between consecutive members (a circular arrangement of 7 people has 7 such gaps); placing each officer into a different gap then guarantees no two officers are next to each other.

Seat the other 7 members around the table first:

$$\text{no. of arrangements of the 7 members} = (7-1)! = 6!$$

This creates 7 gaps (one between each pair of adjacent members). Choose 3 of these 7 gaps for the 3 officers, one officer per gap, and arrange the officers:

$$\text{no. of ways to place the officers} = \binom{7}{3} \times 3!$$

$$\begin{aligned} P(\text{no two officers adjacent}) &= \frac{6! \times \binom{7}{3} \times 3!}{9!} \\ &= \frac{720 \times 35 \times 6}{362880} \\ &= \frac{5}{12} \end{aligned}$$

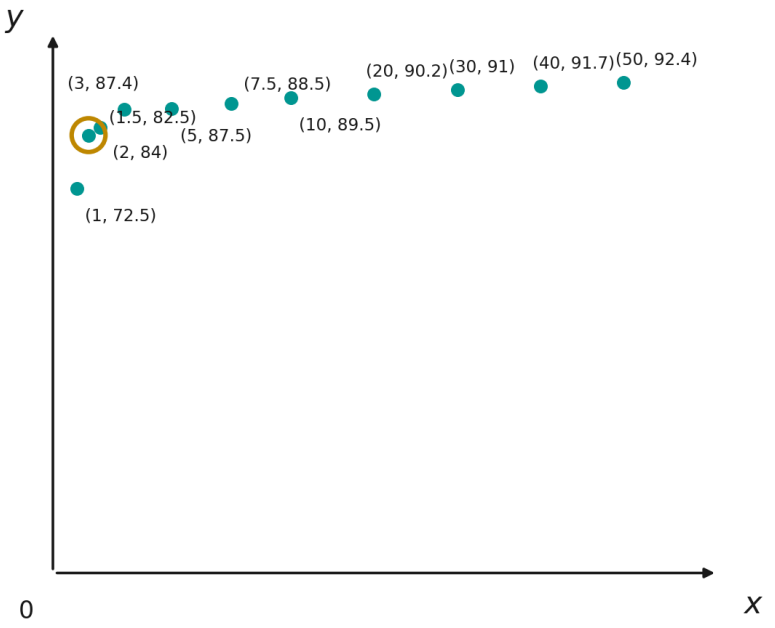
To check your answer: $P(\text{adjacent}) + P(\text{exactly one other person between some pair, etc.}) + P(\text{all separated}) = 1$; here $\frac{1}{12} + \frac{5}{12} = \frac{1}{2}$, leaving $\frac{1}{2}$ for the remaining case (exactly one pair adjacent, the third officer elsewhere), which is a sensible split. ✓

$\therefore\ P(\text{chairperson, secretary and treasurer all separated by at least one other person}) = \frac{5}{12}$.

Question 8 11 marks

Correlation and regression · regression lines

Part (i) 2 marks



The circled point is $(1.5,\ 82.5)$, which breaks the otherwise smooth increasing pattern of the data.

Part (ii) 1 mark

Explanation: the scatter diagram shows the points rising steeply at first and then levelling off towards a roughly horizontal spread as x increases, i.e. the relationship is curved (concave), not a straight line. A model $y = ax + b$ would give a constant rate of increase in y for a fixed increase in x, which does not match the way the points bend over for large x.

Part (iii) 5 marks

Since y increases as x increases (efficiency rises with power), we need $\frac{dy}{dx} > 0$.

$$y = \frac{c}{x} + d$$

$$\frac{dy}{dx} = -\frac{c}{x^2}$$

For $\frac{dy}{dx} > 0$ we need $-c > 0$, i.e. $c < 0$.

As $x \to \infty$, $\frac{c}{x} \to 0$, so $y \to d$. Since the efficiency values level off at a positive percentage (around 90%–92%) rather than at 0 or below, $d > 0$.

$\therefore\ c < 0$ and $d > 0$.

Part (iv) 3 marks

APPROACH

the model $y = \frac{c}{x} + d$ is linear in $u = \frac{1}{x}$, so we let $u = \frac{1}{x}$ and regress y on u, excluding the point $(1.5,\ 82.5)$ identified in part **(i)**.

Let $u = \frac{1}{x}$. Excluding $x = 1.5$, the data become

x	1	2	3	5	7.5	10	20	30	40	50
u	1	0.5	0.333	0.2	0.133	0.1	0.05	0.0333	0.025	0.02
y	72.5	84.0	87.4	87.5	88.5	89.5	90.2	91.0	91.7	92.4

By GC, the regression line of y on u is

$$y = -18.5u + 91.9 \text{ (by GC)}$$

so $c = -18.5$ and $d = 91.9$, both correct to 3 s.f.

$$r = -0.986 \text{ (by GC)}$$

$\therefore r = -0.986$, $c = -18.5$ and $d = 91.9$, all correct to 3 s.f.

Part (v)

Using the model with $x = 1.5$,

$$y = \frac{-18.5}{1.5} + 91.9$$

$$= 79.6 \text{ (3 s.f.)}$$

$\therefore$ the estimated efficiency value that Xian copied wrongly is 79.6%.

Explanation: this estimate is expected to be reliable for two reasons. First, $x = 1.5$ lies within the range of x-values ($1 \leq x \leq 50$) used to obtain the model, so this is interpolation, not extrapolation. Second, the value of $r = -0.986$ found in part **(iv)** is very close to -1, showing that y and $u = \frac{1}{x}$ are very strongly (linearly) correlated, so the model fits the remaining data closely.

Question 9 10 marks

Normal and binomial distributions

Part (a) 2 marks

Since $X \sim N\left(15,\ a^2\right)$, the distribution is symmetric about $\mu = 15$, and 10 and 20 are symmetric about 15 (each 5 away from the mean). Since $P(10 < X < 20) = 0.5$, the two remaining tails together have probability 0.5, split equally by symmetry:

$$P(X < 10) = P(X > 20) = \frac{1 - 0.5}{2} = 0.25$$

so

$$P(X < 20) = 1 - 0.25 = 0.75.$$

Standardising, $Z = \frac{X-15}{a} \sim N(0, 1)$, and

$$P\left(Z < \frac{20 - 15}{a}\right) = 0.75$$

$$\frac{5}{a} = 0.674 \text{ (by GC, invNorm}(0.75))$$

$$a = \frac{5}{0.6744897\ldots}$$

$$= 7.41 \text{ (3 s.f.)}$$

$\therefore a = 7.41.$

Part (b) 4 marks

> **APPROACH**
>
> $P(Y = 1) + P(Y = 2)$ expands to a polynomial in p; we expand and collect it to reach the given quartic, then solve the quartic by GC and reject any root outside $0 < p < 1$ (since p is a probability).

Let $Y \sim B(4, p)$. Then

$$P(Y = 1) = \binom{4}{1} p(1-p)^3 = 4p(1-p)^3$$

$$P(Y = 2) = \binom{4}{2} p^2(1-p)^2 = 6p^2(1-p)^2.$$

Given $P(Y=1)+P(Y=2)=0.5$,

$$\begin{aligned}&4p(1-p)^3+6p^2(1-p)^2\\&=4p\left(1-3p+3p^2-p^3\right)+6p^2\left(1-2p+p^2\right)\\&=4p-12p^2+12p^3-4p^4+6p^2-12p^3+6p^4\\&=2p^4-6p^2+4p\\&=0.5.\end{aligned}$$

Multiplying both sides by 2,

$$4p^4-12p^2+8p=1 \text{ (shown)}$$

Rearranging and solving $4p^4-12p^2+8p-1=0$ by GC,

$$p=0.166,\ 0.599,\ 1.25 \text{ (rej. as } p>1),\ -2.01 \text{ (rej. as } p<0)$$

all values by GC, the last two rejected since $0<p<1$.

$\therefore\ p=0.166$ or $p=0.599$ (3 s.f.).

Part (c) 4 marks

APPROACH

each question has probability $\frac{1}{3}$ of being guessed correctly, and George answers 100 questions independently, so the number correct is $B\left(100,\ \frac{1}{3}\right)$. With $np=33.3>5$ and $n(1-p)=66.7>5$, we approximate this by a normal distribution, applying a continuity correction since we are approximating a discrete distribution by a continuous one.

Let W be the number of questions George guesses correctly, out of 100. Then

$$W\sim B\left(100,\ \frac{1}{3}\right).$$

Since $np=100\times\frac{1}{3}=33.3>5$ and $n(1-p)=100\times\frac{2}{3}=66.7>5$, we use the approximation

$$W\sim N(33.3,\ 22.2) \text{ approximately,}$$

where $E(W)=33.3$ and $Var(W)=100\times\frac{1}{3}\times\frac{2}{3}=22.2$ (to 3 s.f.).

Applying a continuity correction (since W is discrete),

$$\begin{aligned}&P(W\geq 30)\\&\approx P(W>29.5)\\&=0.792 \text{ (3 s.f., by GC)}.\end{aligned}$$

To check your answer: the exact binomial value is $P(W\geq 30)=0.791$ (3 s.f., by GC), very close to the normal approximation 0.792. ✓

$\therefore\ P\left(\text{George guesses at least 30 questions right}\right)=0.792.$

Question 10 14 marks

Poisson distribution · approximations

Part (i) 2 marks

Two conditions needed, in context:

1. Dandelion plants occur independently of one another, i.e. a dandelion plant growing at one point in the field does not affect the probability of a dandelion plant growing at any other point in the field.
2. Dandelion plants occur singly, at random, and at a constant average rate throughout the field, i.e. the mean number of dandelion plants per unit area does not vary from one part of the field to another.

Part (ii) 2 marks

Let X be the number of dandelion plants in a randomly chosen $1\ \text{m}^2$ of Mia's field. Since the average number of dandelions per m^2 is 1.5,

$$X \sim Po(1.5).$$

We want $P(X \geq 2)$. Converting to cumulative form,

$$\begin{aligned} & P(X \geq 2) \\ \Rightarrow\ & 1 - P(X \leq 1) \\ = \ & 1 - 0.558 \\ = \ & 0.442 \ (3\ \text{s.f., by GC}). \end{aligned}$$

$\therefore\ P(X \geq 2) = 0.442.$

Part (iii) 2 marks

Let Y be the number of dandelion plants in a randomly chosen $4\ \text{m}^2$ of Mia's field. Since the numbers of dandelions in disjoint areas are independent, the sum of 4 independent $Po(1.5)$ variables is itself Poisson, with mean 4×1.5:

$$Y \sim Po(6).$$

$$P(Y \leq 3) = 0.151 \ (3\ \text{s.f., by GC}).$$

$\therefore\ P(Y \leq 3) = 0.151.$

Part (iv) 4 marks

APPROACH

the number of dandelions in $80\ \text{m}^2$ is Poisson with a large mean ($\lambda = 120 > 10$), so we approximate it by a normal distribution, using a continuity correction since we are approximating a discrete distribution by a continuous one.

Let W be the number of dandelion plants in an $80\ \text{m}^2$ area of the field. Then

$$W \sim Po(120),$$

since the mean number of dandelions is $1.5 \times 80 = 120$.

Since $\lambda = 120 > 10$ is large, we use the approximation

$$W \sim N(120,\ 120) \text{ approximately},$$

as $E(W) = Var(W) = \lambda = 120$ for a Poisson variable.

Applying a continuity correction (since W is discrete),

$$\begin{aligned} &P(110 \leq W \leq 140) \\ &\approx P(109.5 < W < 140.5) \\ &= 0.800 \text{ (3 s.f., by GC)}. \end{aligned}$$

To check your answer: the exact Poisson value is $P(110 \leq W \leq 140) = 0.798$ (3 s.f., by GC), close to the normal approximation 0.800. ✓

$\therefore\ P(110 \leq W \leq 140) = 0.800.$

Part (v) 4 marks

APPROACH

let N_1 be the number of daisies in $1\ \text{m}^2$ and N_2 the number in $2\ \text{m}^2$, so $N_2 \sim Po(2\lambda)$. The given condition equates $P(N_1 \leq 2)$ with $P(N_2 > 2)$; writing both sides out in terms of λ gives a transcendental equation in λ (it cannot be rearranged algebraically, since λ appears both inside and outside an exponential), so we solve it numerically by GC.

Let $N_1 \sim Po(\lambda)$ be the number of daisies in $1\ \text{m}^2$ of the field, and $N_2 \sim Po(2\lambda)$ be the number of daisies in $2\ \text{m}^2$ of the field. We are given $P(N_1 \leq 2) = P(N_2 > 2)$, i.e.

$$e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2}\right) = 1 - e^{-2\lambda}\left(1 + 2\lambda + 2\lambda^2\right).$$

Solving this equation for $\lambda > 0$ by GC,

$$\lambda = 1.85 \text{ (3 s.f.)}.$$

$\therefore\ \lambda = 1.85.$