TEN-YEAR SERIES · WORKED SOLUTIONS

2016 A-Level H2 Mathematics Paper 1

11 questions · 100 marks

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Physics. Math. Chemistry.

Question 1 5 marks

Inequalities · rational inequalities

We combine the two terms over the common denominator $x-4$.

$$\begin{aligned}&\frac{4x^2+4x-14}{x-4}-(x+3)\\&=\frac{4x^2+4x-14-(x+3)(x-4)}{x-4}\\&=\frac{4x^2+4x-14-\left(x^2-x-12\right)}{x-4}\\&=\frac{3x^2+5x-2}{x-4}\end{aligned}$$

Factorising the numerator,

$$3x^2+5x-2=(3x-1)(x+2)$$

$$\therefore\ \frac{4x^2+4x-14}{x-4}-(x+3)=\frac{(3x-1)(x+2)}{x-4}$$

APPROACH

The inequality $\frac{4x^2+4x-14}{x-4}<x+3$ is exactly the statement that the expression found in part (i) is negative. This avoids cross-multiplying by $x-4$, whose sign depends on x.

Since

$$\frac{4x^2+4x-14}{x-4}-(x+3)=\frac{(3x-1)(x+2)}{x-4},$$

the inequality $\frac{4x^2+4x-14}{x-4}<x+3$ is equivalent to

$$\frac{(3x-1)(x+2)}{x-4}<0.$$

The critical values are $x=-2,\ \frac{1}{3},\ 4$, where $x=4$ is excluded from the domain. We consider the sign of the expression on each interval:

$$\begin{aligned}x<-2:&\ \ (-)(-)/(-)<0\\-2<x<\frac{1}{3}:&\ \ (-)(+)/(-)>0\\\frac{1}{3}<x<4:&\ \ (+)(+)/(-)<0\\x>4:&\ \ (+)(+)/(+)>0\end{aligned}$$

To check your answer: at $x=1$, the left side of the original inequality is $\frac{4+4-14}{1-4}=\frac{-6}{-3}=2$ and $x+3=4$, and indeed $2<4$. ✓

$\therefore x<-2$ or $\frac{1}{3}<x<4$.

Question 2 5 marks

Differentiation · tangents

Part (i) 2 marks

Since $y=2^{\cos x}$,

$$\frac{dy}{dx} = 2^{\cos x} \ln 2 \cdot (-\sin x)$$

By GC,

$$\left.\frac{dy}{dx}\right|_{x=0} = 2^1 \ln 2 \ (-\sin 0) = 0$$

$$\left.\frac{dy}{dx}\right|_{x=\frac{1}{2}\pi} = 2^0 \ln 2 \left(-\sin \frac{1}{2}\pi\right) = -\ln 2 \approx -0.693$$

$\therefore$ the gradient at $x = 0$ is 0, and the gradient at $x = \frac{1}{2}\pi$ is $-\ln 2 \approx -0.693$.

Part (ii) 3 marks

APPROACH

With both gradients and both points known, each tangent is written down directly from $y - y_1 = m\,(x - x_1)$; the two lines are then solved simultaneously for their point of intersection.

At $x = 0$, $y = 2^{\cos 0} = 2^1 = 2$, and the gradient there is 0, so the tangent is horizontal:

$$y = 2$$

At $x = \frac{1}{2}\pi$, $y = 2^{\cos \frac{1}{2}\pi} = 2^0 = 1$, and the gradient there is $-\ln 2$, so the tangent is

$$y - 1 = -\ln 2 \left(x - \frac{1}{2}\pi\right)$$

$$y = 1 - \ln 2 \left(x - \frac{1}{2}\pi\right)$$

To find where the two tangents meet, we substitute $y = 2$ into the second tangent:

$$2 = 1 - \ln 2 \left(x - \frac{1}{2}\pi\right)$$

$$1 = -\ln 2 \left(x - \frac{1}{2}\pi\right)$$

$$x - \frac{1}{2}\pi = -\frac{1}{\ln 2}$$

$$x = \frac{1}{2}\pi - \frac{1}{\ln 2}$$

By GC, $x \approx 0.128$.

To check your answer: substituting $x = \frac{1}{2}\pi - \frac{1}{\ln 2} \approx 0.128$ into $y = 1 - \ln 2\,(x - \frac{1}{2}\pi)$ gives $y \approx 1 - \ln 2\,(-1/\ln 2) = 1 + 1 = 2$, matching the first tangent. ✓

$\therefore$ the tangents are $y = 2$ and $y = 1 - \ln 2\,(x - \frac{1}{2}\pi)$, meeting at the point $(\frac{1}{2}\pi - \frac{1}{\ln 2},\ 2) \approx (0.128,\ 2)$.

Question 3 6 marks

Graphs and transformations · sequence of transformations

APPROACH

The form $k(x - l)^4 + m$ is x^4 translated so that its turning point $(0, 0)$ moves to (l, m), then stretched vertically by factor k. Since the turning point of x^4 corresponds to (a, b) on $y = f(x)$, matching turning points gives l and m directly; k then follows from the second given point $(0, c)$.

The turning point of $y = x^4$ is $(0, 0)$, so under the transformation to $y = k(x - l)^4 + m$ this point becomes (l, m). Since this corresponds to (a, b),

$$l = a, \quad m = b$$

Since $y = f(x)$ also passes through $(0, c)$,

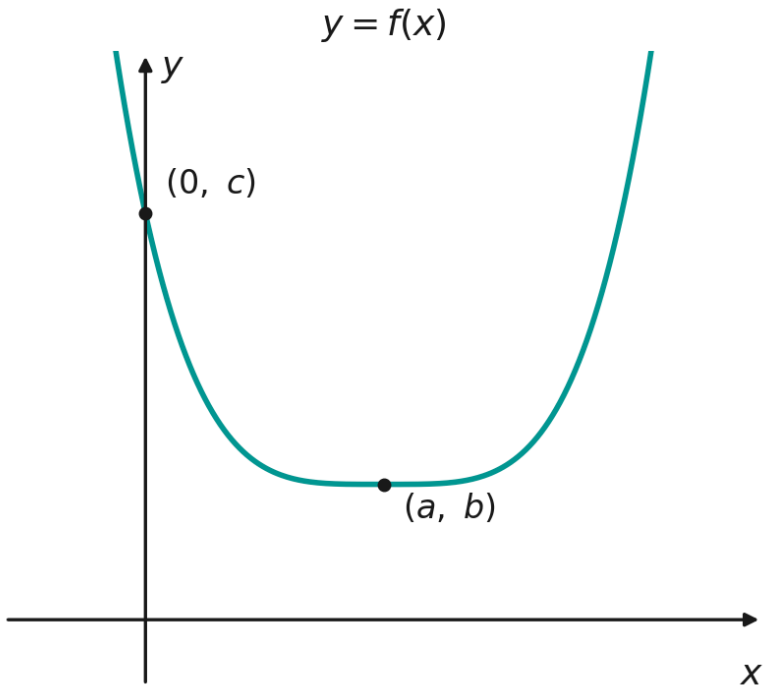
$$f(0) = k(0 - a)^4 + b = c$$
$$ka^4 = c - b$$
$$k = \frac{c - b}{a^4}$$

$\therefore k = \frac{c-b}{a^4}$, $l = a$, $m = b$.

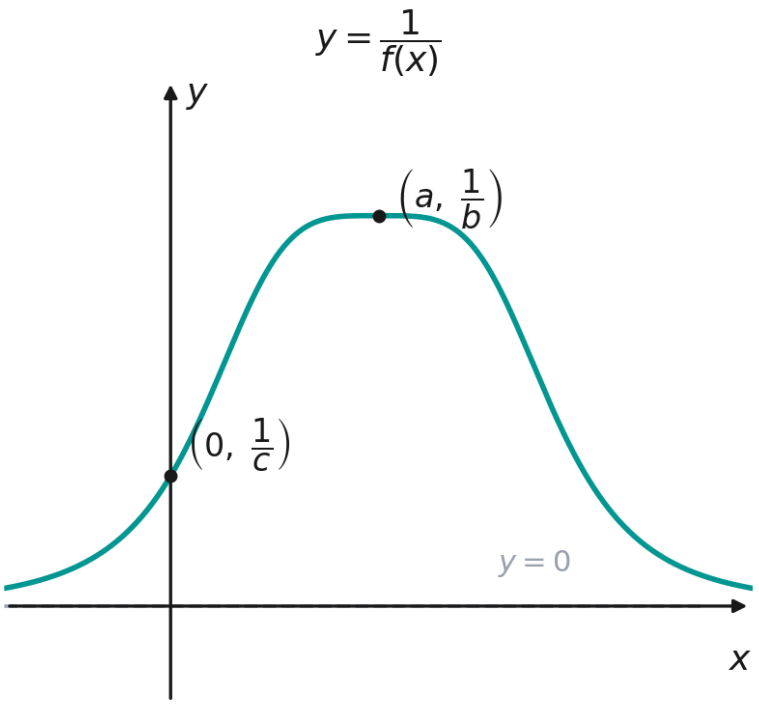
APPROACH

Since a, b, c are positive with $c > b$, we have $k > 0$, so $y = f(x)$ is an upward-opening quartic with minimum value $b > 0$ at $x = a$ — it lies entirely above the x-axis. Reciprocating a curve that stays strictly positive and tends to ∞ turns its minimum into a maximum of $\frac{1}{f(x)}$, sends the large values of $|f(x)|$ to values near 0, and never crosses the x-axis (since $f(x)$ is never 0).

We first sketch $y = f(x)$: a quartic with minimum turning point (a, b) and y-intercept $(0, c)$, where $c > b > 0$ so the curve never meets the x-axis.



Since $f(x) \geq b > 0$ for all x, the curve $y = \frac{1}{f(x)}$ is defined for all x and is always positive. As $f(x) \to \infty$ when $x \to \pm\infty$, we get $\frac{1}{f(x)} \to 0^+$, so $y = 0$ is a horizontal asymptote. The minimum of $f(x)$ at $x = a$ becomes a maximum of $\frac{1}{f(x)}$ there, of value $\frac{1}{b}$. The y-intercept becomes $\frac{1}{f(0)} = \frac{1}{c}$. Since $f(x)$ is never 0, $y = \frac{1}{f(x)}$ never crosses the x-axis.



$\therefore y = \frac{1}{f(x)}$ crosses the y-axis at $\left(0,\ \frac{1}{c}\right)$, does not cross the x-axis, and has a maximum turning point at $\left(a,\ \frac{1}{b}\right)$.

Question 4 7 marks

Sequences and series · AP and GP

Part (i) 4 marks

APPROACH

Four unknowns a, d, b, r appear but only their relationship matters. Writing the three given equalities and subtracting consecutive pairs eliminates a, leaving two equations in d alone; dividing those two equations eliminates d as well, leaving a single equation in r.

Let T_n denote the nth term of the arithmetic series and G_n the nth term of the geometric series. The given information is

$$T_4 = G_5,\ \ T_9 = G_8,\ \ T_{12} = G_{15}$$

i.e.

$$a + 3d = br^4 \qquad (1)$$
$$a + 8d = br^7 \qquad (2)$$
$$a + 11d = br^{14} \qquad (3)$$

Subtracting (1) from (2), and (2) from (3),

$$5d = br^4\left(r^3 - 1\right) \qquad (4)$$
$$3d = br^7\left(r^7 - 1\right) \qquad (5)$$

Since $d \neq 0$, dividing (4) by (5) eliminates d:

$$\frac{5}{3} = \frac{br^4\left(r^3-1\right)}{br^7\left(r^7-1\right)} = \frac{r^3-1}{r^3\left(r^7-1\right)}$$

Cross-multiplying,

$$5r^3\left(r^7-1\right) = 3\left(r^3-1\right)$$
$$5r^{10} - 5r^3 = 3r^3 - 3$$
$$5r^{10} - 8r^3 + 3 = 0 \ \ (\text{shown})$$

Note: $r = 1$ satisfies this equation exactly, but it is excluded by the given condition $|r| < 1$.

By GC, the only root with $|r| < 1$ is

$$r \approx 0.74$$

To check your answer: $5(0.7404)^{10} - 8(0.7404)^3 + 3 \approx 0$, using the unrounded GC value. ✓

$\therefore r \approx 0.74$ (2 d.p.).

Part (ii) 3 marks

APPROACH

"The terms after, but not including, the nth term" is the sum of the geometric series from the $(n+1)$th term onward. Since $|r| < 1$, this tail sum converges, and it equals the $(n+1)$th term divided by $1 - r$.

The $(n+1)$th term of the geometric series is br^n, so the sum of all terms after the nth term is the sum to infinity of a geometric series with first term br^n and common ratio r:

$$\sum_{k=n+1}^{\infty} G_k = \frac{br^n}{1-r}$$

Using $r = 0.74$,

$$\frac{br^n}{1-r} = \frac{b(0.74)^n}{1-0.74} = \frac{b(0.74)^n}{0.26}$$

Simplifying the constant,

$$\frac{1}{0.26} = \frac{50}{13}$$
$$\therefore \sum_{k=n+1}^{\infty} G_k = \frac{50b(0.74)^n}{13}$$

Question 5 8 marks

Vectors · scalar and vector products

Part (i) 6 marks

APPROACH

Expanding $(\mathbf{u}+\mathbf{v}) \times (\mathbf{u}-\mathbf{v})$ directly by the distributive law is quicker than computing $\mathbf{u} \times \mathbf{v}$ first and doubling it, since the cross product is not commutative and the signs are easy to drop; we do it directly with components here.

Since $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = a\mathbf{i} + b\mathbf{k}$,

$$\mathbf{u}+\mathbf{v} = (2+a)\mathbf{i} - \mathbf{j} + (2+b)\mathbf{k}, \qquad \mathbf{u}-\mathbf{v} = (2-a)\mathbf{i} - \mathbf{j} + (2-b)\mathbf{k}.$$

Taking the cross product, with $\mathbf{u}+\mathbf{v}=(2+a)\mathbf{i}-\mathbf{j}+(2+b)\mathbf{k}$ and $\mathbf{u}-\mathbf{v}=(2-a)\mathbf{i}-\mathbf{j}+(2-b)\mathbf{k}$, each component of the cross product is found from the other two components of the two vectors:

$$\mathbf{i}\text{-component}: (-1)(2-b)-(2+b)(-1)=-2+b+2+b=2b$$
$$\mathbf{j}\text{-component}: -\left[(2+a)(2-b)-(2+b)(2-a)\right]=-(-4a+4b)=4a-4b$$
$$\mathbf{k}\text{-component}: (2+a)(-1)-(-1)(2-a)=-2-a+2-a=-2a$$
$$\therefore\ (\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v})=2b\mathbf{i}+(4a-4b)\mathbf{j}-2a\mathbf{k}.$$

Part (ii)

Using the given probability, the $\mathbf{i}$- and $\mathbf{k}$-components of the part (i) answer are equal:

$$2b=-2a$$
$$b=-a$$

Substituting $b=-a$ into the part (i) answer,

$$(\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v})=2(-a)\mathbf{i}+(4a-4(-a))\,\mathbf{j}-2a\mathbf{k}$$
$$\therefore\ (\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v})=-2a\mathbf{i}+8a\mathbf{j}-2a\mathbf{k}.$$

APPROACH

For this vector to be a unit vector its magnitude must equal 1. Since every component is a multiple of a, factor a out of the magnitude first.

$$|(\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v})|=\sqrt{(-2a)^2+(8a)^2+(-2a)^2}=\sqrt{72a^2}=6\sqrt{2}\,|a|$$

Setting this equal to 1,

$$6\sqrt{2}\,|a|=1$$
$$|a|=\frac{1}{6\sqrt{2}}=\frac{\sqrt{2}}{12}$$

To check your answer: with $a=\frac{\sqrt{2}}{12}$, $6\sqrt{2}\cdot\frac{\sqrt{2}}{12}=\frac{12}{12}=1$. ✓

$\therefore a=\frac{\sqrt{2}}{12}$ or $a=-\frac{\sqrt{2}}{12}$.

Part (iii) 2 marks

APPROACH

Expanding $(\mathbf{u}+\mathbf{v})\,.\,(\mathbf{u}-\mathbf{v})$ by the distributive law of the dot product, the cross terms $-\mathbf{u}.\mathbf{v}$ and $+\mathbf{v}.\mathbf{u}$ cancel since the dot product is commutative, leaving a relation between $|\mathbf{u}|^2$ and $|\mathbf{v}|^2$ only — so the individual values of a and b are never needed.

$$(\mathbf{u}+\mathbf{v})\,.\,(\mathbf{u}-\mathbf{v})$$
$$=\mathbf{u}.\mathbf{u}-\mathbf{u}.\mathbf{v}+\mathbf{v}.\mathbf{u}-\mathbf{v}.\mathbf{v}$$
$$=|\mathbf{u}|^2-|\mathbf{v}|^2$$

Using the given condition $(\mathbf{u}+\mathbf{v})\,.\,(\mathbf{u}-\mathbf{v})=0$,

$$|\mathbf{u}|^2-|\mathbf{v}|^2=0$$
$$|\mathbf{v}|^2=|\mathbf{u}|^2$$

Since $|\mathbf{u}|^2=2^2+(-1)^2+2^2=9$,

$$|\mathbf{v}|^2=9$$

As $|\mathbf{v}|$ is a length, $|\mathbf{v}|>0$, so we reject the negative root.

$\therefore |\mathbf{v}| = 3.$

Question 6 10 marks

Sequences and series · induction · recurrence relations

Part (i) 5 marks

Let $P(n)$ be the statement

$$\sum_{r=1}^{n} r\left(r^2+1\right) = \frac{1}{4}n(n+1)\left(n^2+n+2\right).$$

When $n=1$: LHS $=1\left(1^2+1\right)=2$. RHS $=\frac{1}{4}(1)(2)(1+1+2)=\frac{1}{4}(2)(4)=2$. Since LHS $=$ RHS, $P(1)$ is true.

Assume $P(k)$ is true for some $k \in \mathbb{Z}^+$, i.e.

$$\sum_{r=1}^{k} r\left(r^2+1\right) = \frac{1}{4}k(k+1)\left(k^2+k+2\right).$$

We want to show $P(k+1)$ is true, i.e.

$$\sum_{r=1}^{k+1} r\left(r^2+1\right) = \frac{1}{4}(k+1)(k+2)\left((k+1)^2+(k+1)+2\right).$$

$$\sum_{r=1}^{k+1} r\left(r^2+1\right)$$

$$= \sum_{r=1}^{k} r\left(r^2+1\right) + (k+1)\left((k+1)^2+1\right)$$

$$= \frac{1}{4}k(k+1)\left(k^2+k+2\right) + (k+1)\left((k+1)^2+1\right)$$

Factoring out $(k+1)$,

$$\sum_{r=1}^{k+1} r\left(r^2+1\right) = (k+1)\left[\frac{1}{4}k\left(k^2+k+2\right) + (k+1)^2+1\right]$$

Simplifying the bracket,

$$\frac{1}{4}k\left(k^2+k+2\right) + (k+1)^2+1$$

$$= \frac{1}{4}\left(k^3+k^2+2k\right) + k^2+2k+2$$

$$= \frac{k^3+k^2+2k+4k^2+8k+8}{4}$$

$$= \frac{k^3+5k^2+10k+8}{4}$$

Since $(k+2)\left((k+1)^2+(k+1)+2\right) = (k+2)\left(k^2+3k+4\right) = k^3+5k^2+10k+8$, this bracket equals $\frac{1}{4}(k+2)\left((k+1)^2+(k+1)+2\right)$, so

$$\sum_{r=1}^{k+1} r\left(r^2+1\right) = (k+1)\cdot\frac{1}{4}(k+2)\left((k+1)^2+(k+1)+2\right)$$

$$= \frac{1}{4}(k+1)(k+2)\left((k+1)^2+(k+1)+2\right).$$

This is $P(k+1)$, so $P(k)$ true $\Rightarrow P(k+1)$ true.

Since $P(1)$ is true, and $P(k)$ true $\Rightarrow P(k+1)$ true, by the principle of mathematical induction $P(n)$ is true for all $n \in \mathbb{Z}^+$. (shown)

Part (ii) 2 marks

Using the given recurrence $u_n = u_{n-1} + n^3 + n$ with $u_0 = 2$,

$$u_1 = u_0 + 1^3 + 1 = 2 + 2 = 4$$
$$u_2 = u_1 + 2^3 + 2 = 4 + 10 = 14$$
$$u_3 = u_2 + 3^3 + 3 = 14 + 30 = 44$$

$\therefore u_1 = 4,\ u_2 = 14,\ u_3 = 44.$

Part (iii) 3 marks

APPROACH

The sum $\sum_{r=1}^{n}(u_r - u_{r-1})$ telescopes to $u_n - u_0$, since every interior term cancels. On the other hand, each term $u_r - u_{r-1}$ equals $r^3 + r$ by the given recurrence, and $r^3 + r = r\left(r^2 + 1\right)$, so this sum is exactly the sum evaluated in part (i). Equating the two gives u_n.

Since $u_r - u_{r-1} = r^3 + r$ for $r \geq 1$,

$$\sum_{r=1}^{n}(u_r - u_{r-1}) = \sum_{r=1}^{n}\left(r^3 + r\right) = \sum_{r=1}^{n} r\left(r^2 + 1\right)$$

By part (i), this equals $\frac{1}{4}n(n+1)\left(n^2 + n + 2\right)$.

Also, writing out the sum on the left, all terms cancel except the first and last:

$$\sum_{r=1}^{n}(u_r - u_{r-1}) = (u_1 - u_0) + (u_2 - u_1) + \ldots + (u_n - u_{n-1}) = u_n - u_0$$

Comparing the two,

$$u_n - u_0 = \frac{1}{4}n(n+1)\left(n^2 + n + 2\right)$$

Since $u_0 = 2$,

$$u_n = 2 + \frac{1}{4}n(n+1)\left(n^2 + n + 2\right)$$

To check your answer: at $n = 3$, $u_3 = 2 + \frac{1}{4}(3)(4)(9 + 3 + 2) = 2 + \frac{1}{4}(12)(14) = 2 + 42 = 44$, matching part (ii). ✓

$\therefore u_n = 2 + \frac{1}{4}n(n+1)\left(n^2 + n + 2\right).$

Question 7 10 marks

Complex numbers · roots of equations

Part (a) 5 marks

Substituting $w = -1 + 5i$ into $w^2 + (-1 - 8i)w + (-17 + 7i)$,

$$w^2 = (-1 + 5i)^2 = 1 - 10i + 25i^2 = -24 - 10i$$
$$(-1 - 8i)w = (-1 - 8i)(-1 + 5i) = 1 - 5i + 8i - 40i^2 = 41 + 3i$$

Adding these to the constant term,

$w^2 + (-1 - 8i)w + (-17 + 7i) = (-24 - 10i) + (41 + 3i) + (-17 + 7i) = (-24 + 41 - 17) + (-10 + 3 + 7)i = 0.$

Since this equals 0, $-1 + 5i$ satisfies the equation, so it is a root. (shown)

APPROACH

Since all the coefficients of this quadratic are complex, the roots are not a conjugate pair; instead we use the fact that the sum of the roots of $w^2 + pw + q = 0$ is $-p$, which needs only the coefficient of w, not the constant term.

For $w^2 + (-1 - 8i)w + (-17 + 7i) = 0$, the sum of the two roots is

$$-(-1 - 8i) = 1 + 8i$$

Since one root is $-1 + 5i$, the second root is

$$\begin{aligned}(1 + 8i) - (-1 + 5i)\\ = 1 + 8i + 1 - 5i\\ = 2 + 3i\end{aligned}$$

To check your answer: the product of the roots should equal $-17 + 7i$. Indeed $(-1 + 5i)(2 + 3i) = -2 - 3i + 10i + 15i^2 = -17 + 7i$. ✓

$\therefore$ the second root is $p + iq = 2 + 3i$.

Part (b) 5 marks

APPROACH

All the coefficients of $z^3 - 5z^2 + 16z + k$ are real, so non-real roots occur in conjugate pairs. If $1 + ai$ is a root, so is $1 - ai$, and the third root is real (it cannot pair with anything else). Rather than dividing out the known factor, it is faster here to use the sum and sum-of-products relations between the roots and coefficients directly.

Let the third root be $r \in \mathbb{R}$. Since $1 + ai$ and $1 - ai$ are both roots,

Sum of roots $= 5$:

$$\begin{aligned}(1 + ai) + (1 - ai) + r = 5\\ 2 + r = 5\\ r = 3\end{aligned}$$

Sum of products of pairs of roots $= 16$:

$$\begin{aligned}(1 + ai)(1 - ai) + (1 + ai)(r) + (1 - ai)(r) = 16\\ \left(1 + a^2\right) + r\left[(1 + ai) + (1 - ai)\right] = 16\\ \left(1 + a^2\right) + 2r = 16\end{aligned}$$

Substituting $r = 3$,

$$\begin{aligned}1 + a^2 + 6 = 16\\ a^2 = 9\\ a = 3 \ \ (\text{rej } a = -3 \text{ as } a > 0)\end{aligned}$$

Product of roots $= -k$:

$$\begin{aligned}(1 + ai)(1 - ai)(r) = -k\\ \left(1 + a^2\right) r = -k\end{aligned}$$

Substituting $a = 3$, $r = 3$,

$$\begin{aligned}(1 + 9)(3) = -k\\ 30 = -k\end{aligned}$$

$$k = -30$$

To check your answer: with $k = -30$, the cubic $z^3 - 5z^2 + 16z - 30$ has roots $3,\ 1+3i,\ 1-3i$ (by GC / by direct expansion), matching $r = 3$ and $a = 3$. ✓

$\therefore a = 3$ and $k = -30$.

Question 8 11 marks

Maclaurin series · repeated differentiation

Part (i) 8 marks

Since $f(x) = \tan(ax + b)$, by the chain rule,

$$f'(x) = a\sec^2(ax + b)$$

Using the identity $\sec^2\theta = 1 + \tan^2\theta$,

$$f'(x) = a\left(1 + \tan^2(ax + b)\right)$$

Since $y = f(x) = \tan(ax + b)$,

$$f'(x) = a + ay^2. \ \text{(shown)}$$

APPROACH

Rather than differentiating $\tan(ax + b)$ afresh, we differentiate the shown result $f'(x) = a + ay^2$ using the chain rule, treating y as a function of x and writing $y' = f'(x)$ in terms of y again each time. This keeps every derivative expressed in a and y only, as the question asks.

Differentiating $f'(x) = a + ay^2$ with respect to x,

$$f''(x) = 2ay\,y'$$

Since $y' = f'(x) = a + ay^2 = a\left(1 + y^2\right)$,

$$f''(x) = 2ay \cdot a\left(1 + y^2\right)$$
$$\therefore\ f''(x) = 2a^2y\left(1 + y^2\right).$$

Differentiating again with respect to x,

$$f'''(x) = 2a^2\left(y' + 3y^2y'\right) = 2a^2y'\left(1 + 3y^2\right)$$

Substituting $y' = a\left(1 + y^2\right)$,

$$f'''(x) = 2a^2 \cdot a\left(1 + y^2\right) \cdot \left(1 + 3y^2\right)$$
$$\therefore\ f'''(x) = 2a^3\left(1 + y^2\right)\left(1 + 3y^2\right).$$

Part (ii)

APPROACH

The Maclaurin series needs $f(0), f'(0), f''(0), f'''(0)$. Since $y = f(x)$, evaluating y at $x = 0$ first lets us substitute a single number into the part (i) results instead of re-differentiating.

When $b = \frac{1}{4}\pi$, $y(0) = f(0) = \tan\left(\frac{\pi}{4}\right) = 1$.

Substituting $y = 1$ into the part (i) results,

$$f'(0) = a + a(1)^2 = 2a$$

$$f''(0) = 2a^2(1)\left(1+(1)^2\right) = 4a^2$$
$$f'''(0) = 2a^3\left(1+(1)^2\right)\left(1+3(1)^2\right) = 2a^3(2)(4) = 16a^3$$

Using the Maclaurin series $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$,

$$f(x) = 1 + 2ax + \frac{4a^2}{2!}x^2 + \frac{16a^3}{3!}x^3 + \ldots$$
$$\therefore\ f(x) = 1 + 2ax + 2a^2x^2 + \frac{8}{3}a^3x^3 + \ldots$$

Part (iii) 3 marks

APPROACH

$\tan 2x$ is $f(x)$ of part (i) with $a = 2$ and $b = 0$, so instead of expanding from scratch we substitute these values into the part (i) results, this time with $y(0) = \tan(0) = 0$.

Taking $a = 2$, $b = 0$ in part (i), $y(0) = f(0) = \tan(0) = 0$. Substituting $y = 0$,

$$f'(0) = a + a(0)^2 = a = 2$$
$$f''(0) = 2a^2(0)\left(1+(0)^2\right) = 0$$
$$f'''(0) = 2a^3\left(1+(0)^2\right)\left(1+3(0)^2\right) = 2a^3 = 2(2)^3 = 16$$

Note: $f''(0) = 0$, so the x^2 term vanishes; the first two non-zero terms are the x^1 and x^3 terms.

Using the Maclaurin series with $f(0) = 0$, $f'(0) = 2$, $f''(0) = 0$, $f'''(0) = 16$,

$$\tan 2x = 0 + 2x + \frac{0}{2!}x^2 + \frac{16}{3!}x^3 + \ldots = 2x + \frac{8}{3}x^3 + \ldots$$

To check your answer: by GC, expanding $\tan(2x)$ to this order gives $2x + \frac{8}{3}x^3 + \frac{64}{15}x^5 + \ldots$, matching the first two non-zero terms found here. ✓

$\therefore$ the first two non-zero terms are $2x + \frac{8}{3}x^3$.

Question 9 12 marks

Differential equations · motion in a fluid

Part (i)(a) 1 mark

Since $y = \frac{dx}{dt}$, differentiating with respect to t gives $\frac{dy}{dt} = \frac{d^2x}{dt^2}$. Substituting into the given differential equation:

$$\frac{dy}{dt} + 2y = 10$$
$$\frac{dy}{dt} = 10 - 2y \quad \text{(shown)}$$

Part (i)(b) 6 marks

APPROACH

$\frac{dy}{dt} = 10 - 2y$ has y and t separable, so we integrate directly and use $y = 0$ at $t = 0$ (since $\frac{dx}{dt} = 0$ when $t = 0$) to fix the constant. Once y is known in terms of t, we integrate again — using $x = 0$ at $t = 0$ — to get x.

Since $\frac{dy}{dt} = 10 - 2y$, separating variables,

$$\int \frac{1}{10-2y}\,dy = \int 1\,dt$$
$$-\frac{1}{2}\ln|10-2y| = t + C$$

Using $y = 0$ when $t = 0$:

$$-\frac{1}{2}\ln 10 = C$$

Substituting back,

$$-\frac{1}{2}\ln|10-2y| = t - \frac{1}{2}\ln 10$$
$$\ln|10-2y| = \ln 10 - 2t$$
$$|10-2y| = 10e^{-2t}$$

Note: the stone starts at rest and speeds up towards a terminal value, so y increases from 0 towards 5; $10-2y$ therefore stays positive throughout, and the modulus can be dropped.

$$10-2y = 10e^{-2t}$$
$$y = 5 - 5e^{-2t}$$

So $\frac{dx}{dt} = 5 - 5e^{-2t}$. Integrating with respect to t:

$$x = \int \left(5 - 5e^{-2t}\right) dt$$
$$= 5t + \frac{5}{2}e^{-2t} + D$$

Using $x = 0$ when $t = 0$:

$$0 = \frac{5}{2} + D$$
$$D = -\frac{5}{2}$$
$$x = 5t + \frac{5}{2}e^{-2t} - \frac{5}{2}$$

To check your answer: at $t = 0$, $x = 0 + \frac{5}{2} - \frac{5}{2} = 0$, matching the given initial condition. ✓

$\therefore y = 5 - 5e^{-2t}$ and $x = 5t + \frac{5}{2}e^{-2t} - \frac{5}{2}$.

Part (ii) 3 marks

APPROACH

here $\frac{d^2x}{dt^2}$ is given directly as a function of t, so we integrate twice in succession, using $\frac{dx}{dt} = 0$ at $t = 0$ to fix the first constant and $x = 0$ at $t = 0$ to fix the second.

$$\frac{d^2x}{dt^2} = 10 - 5\sin\frac{1}{2}t$$

Integrating with respect to t:

$$\frac{dx}{dt} = 10t + 10\cos\frac{1}{2}t + C_1$$

Using $\frac{dx}{dt} = 0$ when $t = 0$:

$$0 = 10\cos 0 + C_1$$
$$C_1 = -10$$
$$\frac{dx}{dt} = 10t + 10\cos\frac{1}{2}t - 10$$

Integrating again:

$$x = 5t^2 + 20\sin\frac{1}{2}t - 10t + C_2$$

Using $x = 0$ when $t = 0$:

$$0 = 20\sin 0 + C_2$$
$$C_2 = 0$$
$$x = 5t^2 - 10t + 20\sin\frac{1}{2}t$$

$\therefore x = 5t^2 - 10t + 20\sin\frac{1}{2}t$.

Part (iii) 2 marks

APPROACH

the pond is $5\ m$ deep, so for each model we set $x = 5$ and solve for t. Neither equation can be solved for t algebraically, so we treat each as a GC numerical-solution problem.

For Model (i), setting $x = 5$:

$$5t + \frac{5}{2}e^{-2t} - \frac{5}{2} = 5$$

By GC, $t = 1.47\ s$ (2 d.p.).

For Model (ii), setting $x = 5$:

$$5t^2 - 10t + 20\sin\frac{1}{2}t = 5$$

By GC, $t = 1.05\ s$ (2 d.p.).

$\therefore$ Model (i) predicts the stone reaches the bottom at $t = 1.47\ s$; Model (ii) predicts $t = 1.05\ s$.

Question 10 13 marks

Functions · inverse · composite functions

Part (a)(i) 3 marks

Let $y = f(x) = 1 + \sqrt{x}$, for $x \geq 0$. Then

$$\sqrt{x} = y - 1$$
$$x = (y-1)^2$$

Since $x \geq 0$, this requires $y \geq 1$. Interchanging x and y,

$$f^{-1}(x) = (x-1)^2.$$

The domain of f^{-1} is the range of f. Since $\sqrt{x} \geq 0$ for $x \geq 0$, $f(x) \geq 1$, so the range of f is $x \geq 1$.

$\therefore f^{-1}(x) = (x-1)^2$, for $x \in \mathbb{R}$, $x \geq 1$.

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Part (a)(ii) 5 marks

> **APPROACH**
> we compute $ff(x)$ explicitly and set it equal to x, then eliminate the two nested square roots by isolating and squaring twice in succession. Squaring can introduce extraneous roots, so this only shows the *necessity* of the cubic, which is all the question asks for.

Since $f(x) = 1 + \sqrt{x}$,

$$ff(x) = 1 + \sqrt{1 + \sqrt{x}}$$

Setting $ff(x) = x$:

$$1 + \sqrt{1 + \sqrt{x}} = x$$

$$\sqrt{1 + \sqrt{x}} = x - 1$$

Squaring both sides,

$$1 + \sqrt{x} = (x-1)^2$$

$$\sqrt{x} = x^2 - 2x$$

Squaring again,

$$x = \left(x^2 - 2x\right)^2$$

$$= x^4 - 4x^3 + 4x^2$$

$$x^4 - 4x^3 + 4x^2 - x = 0$$

$$x\left(x^3 - 4x^2 + 4x - 1\right) = 0$$

Note: $x = 0$ does not itself satisfy $ff(x) = x$, since $f(0) = 1$ and $ff(0) = f(1) = 2 \neq 0$; it is an extraneous factor introduced by squaring. Any genuine solution therefore has $x \neq 0$, so it must satisfy the remaining factor:

$$x^3 - 4x^2 + 4x - 1 = 0 \quad \text{(shown)}$$

By inspection, $x = 1$ is a root of this cubic, so it factorises as

$$x^3 - 4x^2 + 4x - 1 = (x-1)\left(x^2 - 3x + 1\right).$$

Solving $x^2 - 3x + 1 = 0$ by the quadratic formula gives $x = \frac{3 \pm \sqrt{5}}{2}$.

Note that $x^2 - 3x + 1 = 0$ is exactly the equation obtained by solving $f(x) = x$ directly: $1 + \sqrt{x} = x$ gives $\sqrt{x} = x - 1$, and squaring gives $x = (x-1)^2$, i.e. $x^2 - 3x + 1 = 0$ (valid since this needs $x - 1 \geq 0$, i.e. $x \geq 1$). Of the two roots, only $x = \frac{3+\sqrt{5}}{2}$ (≈ 2.618) satisfies $x \geq 1$; the other, $x = \frac{3-\sqrt{5}}{2}$ (≈ 0.382), is rejected.

So $x = \frac{3+\sqrt{5}}{2}$ satisfies $f(x) = x$.

Since $f(x) = x$, applying f^{-1} to both sides gives $f^{-1}(f(x)) = f^{-1}(x)$, i.e. $x = f^{-1}(x)$. Combined with $f(x) = x$, this gives $f(x) = x = f^{-1}(x)$.

To check your answer: $f\left(\frac{3+\sqrt{5}}{2}\right) = 1 + \sqrt{\frac{3+\sqrt{5}}{2}}$; since $\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{3+\sqrt{5}}{2}$, this is $1 + \frac{1+\sqrt{5}}{2} = \frac{3+\sqrt{5}}{2}$, matching. ✓

Note: the other candidate, $x = 1$, is rejected, since $f(1) = 2$ but $f^{-1}(1) = 0$, so $f(1) \neq f^{-1}(1)$; it is the extraneous root that entered when we squared to derive the cubic (indeed $ff(1) = f(2) = 1 + \sqrt{2} \neq 1$, so $x = 1$ does not even satisfy $ff(x) = x$).

$\therefore x = \frac{3+\sqrt{5}}{2}$ satisfies $f(x) = f^{-1}(x)$, because it is a fixed point of f (so f^{-1} maps it to itself too).

Part (b)(i) 3 marks

$$g(1) = 1 + g(0) = 1 + 1 = 2$$
$$g(2) = 2 + g(1) = 2 + 2 = 4$$
$$g(4) = 2 + g(2) = 2 + 4 = 6$$
$$g(3) = 1 + g(2) = 1 + 4 = 5$$
$$g(6) = 2 + g(3) = 2 + 5 = 7$$
$$g(7) = 1 + g(6) = 1 + 7 = 8$$
$$g(12) = 2 + g(6) = 2 + 7 = 9$$

$\therefore g(4) = 6$, $g(7) = 8$, $g(12) = 9$.

Part (b)(ii) 2 marks

g does not have an inverse.

Explanation: an inverse exists only if g is one-one. But $g(7) = 8$ and $g(8) = 2 + g(4) = 2 + 6 = 8$, so two different elements of the domain, $n = 7$ and $n = 8$, give the same value $g(n) = 8$. Since g is not one-one, g^{-1} does not exist.

Question 11 13 marks

Vectors · planes · lines

Part (i)(a) 5 marks

APPROACH

to show l is perpendicular to p, we show the direction vector of l is parallel to a normal vector of p. To find the intersection, rather than solving three simultaneous equations in λ, μ, t at once, it is quicker to find the Cartesian equation of p, substitute the parametric point of l into it to get t, and then back out λ and μ from the vector equation of p.

With $a = 0$, p has direction vectors $\begin{pmatrix}1\\2\\0\end{pmatrix}$ and $\begin{pmatrix}0\\4\\-2\end{pmatrix}$. A normal vector of p is

$$\mathbf{n} = \begin{pmatrix}1\\2\\0\end{pmatrix} \times \begin{pmatrix}0\\4\\-2\end{pmatrix} = \begin{pmatrix}-4\\2\\4\end{pmatrix} = 2\begin{pmatrix}-2\\1\\2\end{pmatrix}.$$

Since l has direction vector $\begin{pmatrix}-2\\1\\2\end{pmatrix}$, which is parallel to $\mathbf{n}$, l is perpendicular to p (shown).

Using the point $(1, -3, 2)$ on p and normal $\begin{pmatrix}-2\\1\\2\end{pmatrix}$, the Cartesian equation of p is

$$-2x + y + 2z = -2(1) + (-3) + 2(2)$$
$$= -1.$$

With $a = 0$, l is $x = -1 - 2t$, $y = t$, $z = 1 + 2t$. Substituting into the equation of p:

$$-2(-1 - 2t) + t + 2(1 + 2t) = -1$$
$$4 + 9t = -1$$

$$t = -\frac{5}{9}$$

So the point of intersection has coordinates

$$\left(-1 - 2\left(-\frac{5}{9}\right),\ -\frac{5}{9},\ 1 + 2\left(-\frac{5}{9}\right)\right) = \left(\frac{1}{9},\ -\frac{5}{9},\ -\frac{1}{9}\right).$$

To find λ and μ, we equate this point to the vector equation of p:

$$\begin{pmatrix}1\\-3\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\0\end{pmatrix} + \mu\begin{pmatrix}0\\4\\-2\end{pmatrix} = \begin{pmatrix}\frac{1}{9}\\-\frac{5}{9}\\-\frac{1}{9}\end{pmatrix}.$$

From the x-component, $1 + \lambda = \frac{1}{9}$, so $\lambda = -\frac{8}{9}$.

From the z-component, $2 - 2\mu = -\frac{1}{9}$, so $\mu = \frac{19}{18}$.

To check your answer: the y-component gives $-3 + 2\left(-\frac{8}{9}\right) + 4\left(\frac{19}{18}\right) = -3 - \frac{16}{9} + \frac{38}{9} = -\frac{5}{9}$, matching. ✓

$\therefore \lambda = -\frac{8}{9}$, $\mu = \frac{19}{18}$, $t = -\frac{5}{9}$, and l meets p at $\left(\frac{1}{9},\ -\frac{5}{9},\ -\frac{1}{9}\right)$.

Part (i)(b) 5 marks

APPROACH

planes at perpendicular distance 12 from p are planes parallel to p (same normal), obtained by shifting the constant on the right of the Cartesian equation of p by $\pm 12 \times |\mathbf{n}|$.

p has Cartesian equation $-2x + y + 2z = -1$, with normal $\mathbf{n} = \begin{pmatrix}-2\\1\\2\end{pmatrix}$, $|\mathbf{n}| = \sqrt{4+1+4} = 3$.

A general plane parallel to p has equation $-2x + y + 2z = k$ for some constant k. The perpendicular distance between this plane and p is

$$\frac{|k - (-1)|}{3} = 12$$
$$|k + 1| = 36$$
$$k = 35 \text{ or } k = -37$$

$\therefore$ the planes are $-2x + y + 2z = 35$ and $-2x + y + 2z = -37$.

Part (ii) 3 marks

APPROACH

l fails to meet p in a unique point exactly when l is parallel to p — that is, when the direction vector of l is perpendicular to the normal of p. (This covers both the case where l is disjoint from p, giving no intersection, and the case where l lies entirely in p, giving infinitely many.) We find the normal of p in terms of a and set its dot product with the direction of l to zero.

For general a, a normal vector of p is

$$\begin{pmatrix}1\\2\\0\end{pmatrix} \times \begin{pmatrix}a\\4\\-2\end{pmatrix} = \begin{pmatrix}-4\\2\\4-2a\end{pmatrix}.$$

l has direction vector $\begin{pmatrix}-2\\1\\2\end{pmatrix}$. Requiring this to be perpendicular to the normal of p:

$$\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 2 \\ 4-2a \end{pmatrix} = 0$$

$$8 + 2 + 2(4 - 2a) = 0$$

$$18 - 4a = 0$$

$$a = \frac{9}{2}$$

$\therefore a = \frac{9}{2}$.

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