TEN-YEAR SERIES · WORKED SOLUTIONS

2015 A-Level H2 Mathematics Paper 1

11 questions · 100 marks

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Physics. Math. Chemistry.

Question 1 7 marks

Equations · simultaneous equations from a curve

Part (i) 4 marks

Since C passes through $(1.6,\ -2.4)$,

$$\frac{a}{1.6^2} + 1.6b + c = -2.4$$
$$\Rightarrow 0.390625a + 1.6b + c = -2.4. \quad (1)$$

Since C passes through $(-0.7,\ 3.6)$,

$$\frac{a}{(-0.7)^2} - 0.7b + c = 3.6$$
$$\Rightarrow 2.040816a - 0.7b + c = 3.6. \quad (2)$$

Differentiating,

$$\frac{dy}{dx} = -\frac{2a}{x^3} + b.$$

Using the given gradient of 2 at $x = 1$,

$$-2a + b = 2. \quad (3)$$

By GC, solving (1), (2) and (3) simultaneously,

$$a = -3.593,\ \ b = -5.187,\ \ c = 7.303\ \ (3\text{ d.p.}).$$

Part (ii) 2 marks

Approach: We use the unrounded values of a, b, c stored on the GC (not the values rounded to 3 d.p.) to find where C meets the x-axis, then round the final answer only.

C crosses the x-axis where $y = 0$, i.e.

$$\frac{a}{x^2} + bx + c = 0.$$

By GC, solving this numerically with the stored values of a, b, c, the only real root is

$$x = -0.589\ \ (3\text{ d.p.}).$$

To check your answer: substituting $x = -0.589$ into $y = \frac{a}{x^2} + bx + c$ with the unrounded constants gives $y \approx 0$. ✓

$\therefore C$ crosses the x-axis at $x = -0.589$.

Part (iii) 1 mark

As $x \to \pm\infty$, $\frac{a}{x^2} \to 0$, so $y \to bx + c$.

$\therefore$ the other asymptote is $y = bx + c$, i.e. $y = -5.187x + 7.303$.

Question 2 6 marks

Graphs · modulus · asymptotes

Part (i) 3 marks

Approach: Write $f(x) = \frac{x+1}{1-x}$ so that the required curve is $y = |f(x)|$. It is easiest to sketch $y = f(x)$ first and then reflect the part below the x-axis.

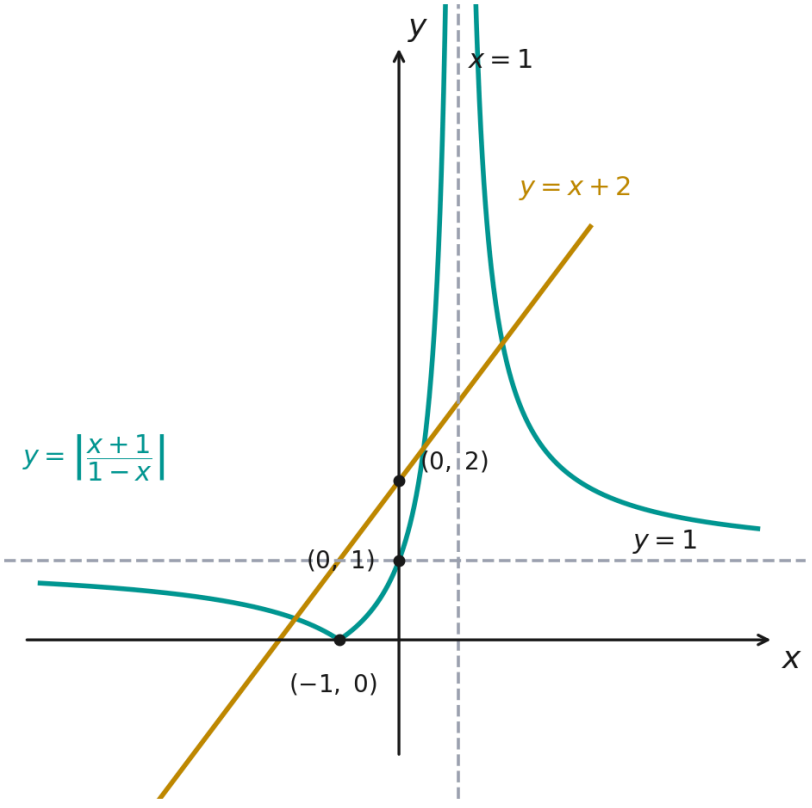
Since f is undefined at $x = 1$, there is a vertical asymptote $x = 1$.

$$\begin{aligned} f(x) &= \frac{x+1}{1-x} \\ &= \frac{-(x-1)+2}{-(x-1)} \\ &= -1 - \frac{2}{x-1}. \end{aligned}$$

As $x \to \pm\infty$, $\frac{2}{x-1} \to 0$, so $f(x) \to -1$: the horizontal asymptote of f is $y = -1$, so for $y = |f(x)|$ the horizontal asymptote is $y = 1$.

$f(x) = 0$ at $x = -1$, so $y = |f(x)|$ meets the x-axis at $(-1,\ 0)$, and $y = |f(0)| = 1$ gives the y-intercept $(0,\ 1)$.

Note: $|f(x)|$ has a sharp corner (not a smooth turning point) at $(-1,\ 0)$, since this is where the expression inside the modulus changes sign.



Part (ii) 3 marks

Approach: $\left|\frac{x+1}{1-x}\right| < x+2$ is exactly the inequality $|f(x)| <$ (line), so its solution is read off the sketch in (i) as the set of x for which the curve lies strictly below the line — provided we also solve algebraically for the exact intersection points, since a sketch alone does not give exact values.

Since the left side is non-negative, we need $x+2>0$, i.e. $x>-2$, for the inequality to have any chance of holding.

We find where the curve meets the line, splitting by the sign of $f(x)$.

For $-1 \le x < 1$, $f(x) \ge 0$, so $|f(x)| = f(x)$:

$$\begin{aligned} \frac{x+1}{1-x} &= x+2 \\ \Rightarrow x+1 &= (x+2)(1-x) \\ \Rightarrow x+1 &= -x^2-x+2 \\ \Rightarrow x^2+2x-1 &= 0 \\ \Rightarrow x &= -1 \pm \sqrt{2}. \end{aligned}$$

Only $x = \sqrt{2}-1 \approx 0.414$ lies in $-1 \le x < 1$; $x = -1-\sqrt{2}$ is rejected as outside this range.

For $x < -1$ or $x > 1$, $f(x) < 0$, so $|f(x)| = -f(x)$:

$$\begin{aligned} -\frac{x+1}{1-x} &= x+2 \\ \Rightarrow -(x+1) &= (x+2)(1-x) \\ \Rightarrow -x-1 &= -x^2-x+2 \\ \Rightarrow x^2 &= 3 \\ \Rightarrow x &= \pm\sqrt{3}. \end{aligned}$$

Both $x = -\sqrt{3} \approx -1.732$ and $x = \sqrt{3} \approx 1.732$ lie in the valid range ($x < -1$ or $x > 1$ respectively), so both are kept.

From the sketch, the curve $y = |f(x)|$ lies below the line $y = x+2$ on $\left(-\sqrt{3},\ \sqrt{2}-1\right)$ (passing continuously through the corner at $x=-1$, where both graphs are defined), and again for $x > \sqrt{3}$, where the line has overtaken the curve beyond the vertical asymptote.

To check your answer: at $x=0$, $\left|\frac{0+1}{1-0}\right| = 1 < 2$ ✓, and 0 lies in $\left(-\sqrt{3},\ \sqrt{2}-1\right)$. ✓

$\therefore -\sqrt{3} < x < \sqrt{2}-1$ or $x > \sqrt{3}$.

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Question 3 5 marks

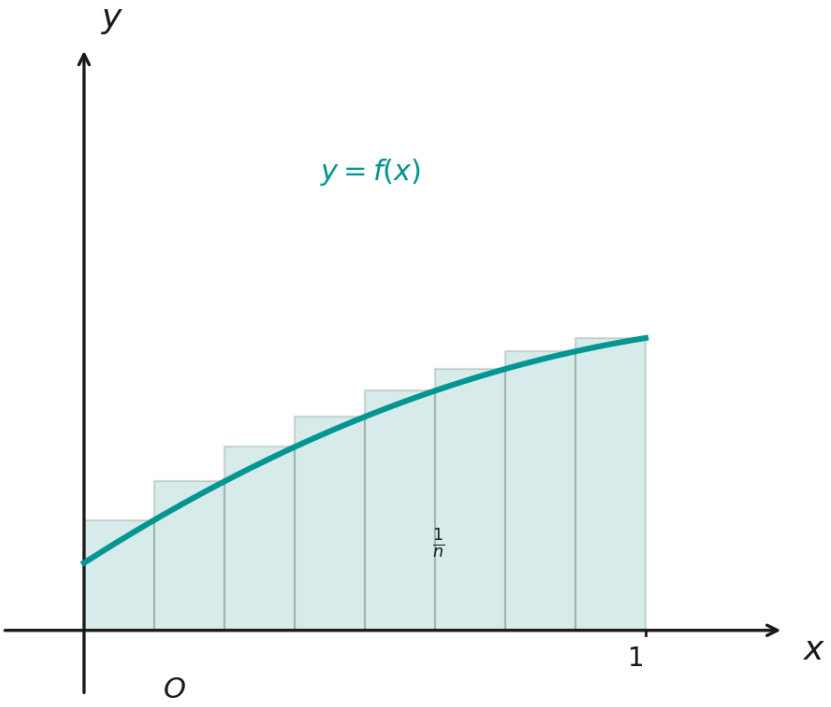
Integration · limit of a sum · definite integrals

Part (i) 2 marks

Approach: Interpret the sum as an area. Split $[0,\ 1]$ into n strips each of width $\frac{1}{n}$, and build a rectangle on each strip whose height is the value of f at the strip's right endpoint.

$$\frac{1}{n}\left\{f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+\ldots+f\left(\frac{n}{n}\right)\right\}$$

is the total area of n rectangles, each of width $\frac{1}{n}$, with the k-th rectangle having height $f\left(\frac{k}{n}\right)$, as shown:



This sum is the total shaded area, which approximates the area under $y=f(x)$ from $x=0$ to $x=1$. As $n\to\infty$, the width $\frac{1}{n}$ of each strip $\to 0$, so the rectangles fit the curve more and more closely, and (since f is continuous, so has no jumps for the rectangles to overshoot or undershoot indefinitely) the total rectangular area converges to the exact area under the curve.

$$\therefore \lim_{n\to\infty}\frac{1}{n}\left\{f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+\ldots+f\left(\frac{n}{n}\right)\right\}=\int_0^1 f(x)\,dx. \quad \text{(shown)}$$

Part (ii) 3 marks

Approach: Write the given limit in the standard form of part (i) by pulling $\sqrt[3]{n}$ inside each term of the sum.

$$\lim_{n\to\infty}\frac{1}{n}\left(\frac{\sqrt[3]{1}+\sqrt[3]{2}+\ldots+\sqrt[3]{n}}{\sqrt[3]{n}}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n} \left\{ \sqrt[3]{\frac{1}{n}} + \sqrt[3]{\frac{2}{n}} + \ldots + \sqrt[3]{\frac{n}{n}} \right\}.$$

This is exactly the sum in (i) with $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$. Hence by (i),

$$\lim_{n\to\infty} \frac{1}{n} \left(\frac{\sqrt[3]{1} + \sqrt[3]{2} + \ldots + \sqrt[3]{n}}{\sqrt[3]{n}} \right) = \int_0^1 x^{\frac{1}{3}}\, dx$$
$$= \left[\frac{3}{4} x^{\frac{4}{3}} \right]_0^1$$
$$= \frac{3}{4}.$$

To check your answer: taking $n = 10^6$ numerically gives $\frac{1}{n}\left(\frac{\sqrt[3]{1}+\ldots+\sqrt[3]{n}}{\sqrt[3]{n}}\right) \approx 0.750000$, matching $\frac{3}{4}$. ✓

$\therefore$ the limit is $\frac{3}{4}$.

Question 4 6 marks

Differentiation · maximum and minimum problems

Approach: Set up the total wire length as a constraint linking x and y, use it to write y in terms of x (and d), then express the total area as a function of x alone and maximise it.

Let the total area of the rectangle and the semicircle be $A\ \mathrm{m}^2$.

The rectangle has perimeter $2x + 2y$. The semicircle, including its diameter, has perimeter (the curved arc plus the straight diameter) $\pi x + 2x$. Since the wire is cut into exactly these two pieces,

$$2x + 2y + \pi x + 2x = d$$
$$\Rightarrow 2y = d - 4x - \pi x$$
$$\Rightarrow y = \frac{d}{2} - 2x - \frac{\pi x}{2}.$$

Since $y > 0$, this requires $0 < x < \frac{d}{4+\pi}$.

The total area is the rectangle's area plus the semicircle's area:

$$A = xy + \frac{1}{2}\pi x^2.$$

Substituting the expression for y,

$$A = x\left(\frac{d}{2} - 2x - \frac{\pi x}{2}\right) + \frac{1}{2}\pi x^2$$
$$= \frac{d}{2}x - 2x^2 - \frac{\pi}{2}x^2 + \frac{\pi}{2}x^2$$
$$= \frac{d}{2}x - 2x^2.$$

Note: the π terms cancel exactly, so A depends on x but not directly on π at this stage.

Differentiating,

$$\frac{dA}{dx} = \frac{d}{2} - 4x.$$

At a stationary point, $\frac{dA}{dx} = 0$:

$$\frac{d}{2} - 4x = 0$$
$$\Rightarrow x = \frac{d}{8}.$$

Since $\frac{d}{8} < \frac{d}{4+\pi}$ (as $8 > 4 + \pi$), this value of x lies in the valid range.

$$\frac{d^2A}{dx^2} = -4 < 0,$$

so A is maximum at $x = \frac{d}{8}$.

Substituting back,

$$A_{\max} = \frac{d}{2}\left(\frac{d}{8}\right) - 2\left(\frac{d}{8}\right)^2$$
$$= \frac{d^2}{16} - \frac{d^2}{32}$$
$$= \frac{d^2}{32}.$$

To check your answer: at $x = \frac{d}{8}$, $y = \frac{d}{2} - 2\left(\frac{d}{8}\right) - \frac{\pi}{2}\left(\frac{d}{8}\right) = \frac{d(4-\pi)}{16} > 0$ since $\pi < 4$, confirming this is a genuine (feasible) maximum. ✓

∴ the maximum total area is $\frac{d^2}{32}\ \text{m}^2$, so $k = \frac{1}{32}$. (shown)

Question 5 7 marks

Graphs and transformations · sequence of transformations

Part (i) 2 marks

APPROACH

Write $\frac{1}{4}(x-3)^2$ as a chain of operations applied to x^2, starting from the inside (the x that gets replaced) and working outwards to the factor multiplying the whole expression.

We first replace x by $x - 3$, which translates the curve 3 units in the positive x-direction:

$$y = x^2 \rightarrow y = (x-3)^2.$$

We then multiply the whole expression by $\frac{1}{4}$, which scales the curve parallel to the y-axis by factor $\frac{1}{4}$:

$$y = (x-3)^2 \rightarrow y = \frac{1}{4}(x-3)^2.$$

∴ the curve $y = x^2$ is translated 3 units in the positive x-direction, then scaled parallel to the y-axis by factor $\frac{1}{4}$.

Part (ii) 3 marks

> **APPROACH**
>
> Before sketching, we check the value of f at each piece boundary and, where a piece is not included at a boundary, note whether the point is actually attained (a jump) or only approached.

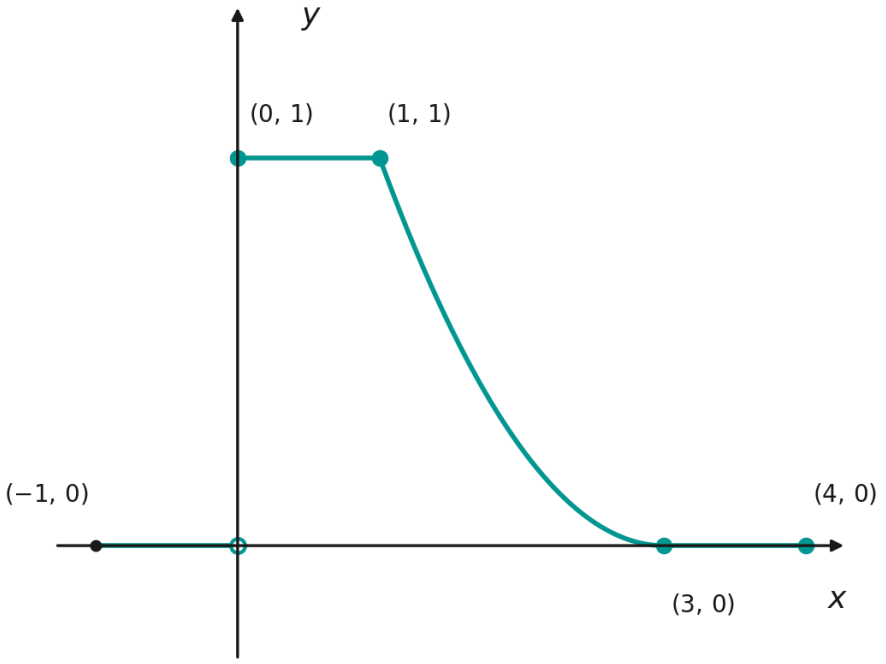
On $0 \leq x \leq 1$, $f(x) = 1$, a horizontal segment from $(0, 1)$ to $(1, 1)$, both endpoints attained.

On $1 < x \leq 3$, $f(x) = \frac{1}{4}(x-3)^2$, part of an upward parabola with vertex $(3, 0)$. At $x = 1$ (not included in this piece, but the value matches the flat segment already found):

$$f(1) = \frac{1}{4}(1-3)^2 = 1.$$

So the curve comes in at $(1, 1)$ with a corner (the gradient jumps from 0 to -1 there), then decreases to $(3, 0)$.

Otherwise, $f(x) = 0$. For $-1 \leq x < 0$ this gives a horizontal segment on the x-axis, but at $x = 0$ itself $f(0) = 1$ (the first piece applies), so there is a jump: f approaches 0 as $x \to 0^-$ but $f(0) = 1$. For $3 < x \leq 4$, $f(x) = 0$, and since $f(3) = 0$ too, the curve meets this segment smoothly (no jump) at $(3, 0)$.



Note: the open circle at $(0, 0)$ marks the value approached but not attained; $(0, 1)$ is the actual value of $f(0)$.

Part (iii) 2 marks

> **APPROACH**
>
> $y = 1 + f\left(\frac{1}{2}x\right)$ is f scaled parallel to the x-axis by factor 2 (replacing x by $\frac{1}{2}x$), then translated 1 unit in the positive y-direction. The three break-points of f at $x = 0, 1, 3$ move to $x = 0, 2, 6$ under the stretch; we only need the part of the new curve on $-1 \le x \le 4$.

For $0 \le \frac{1}{2}x \le 1$, i.e. $0 \le x \le 2$: $f\left(\frac{1}{2}x\right) = 1$, so $y = 2$.

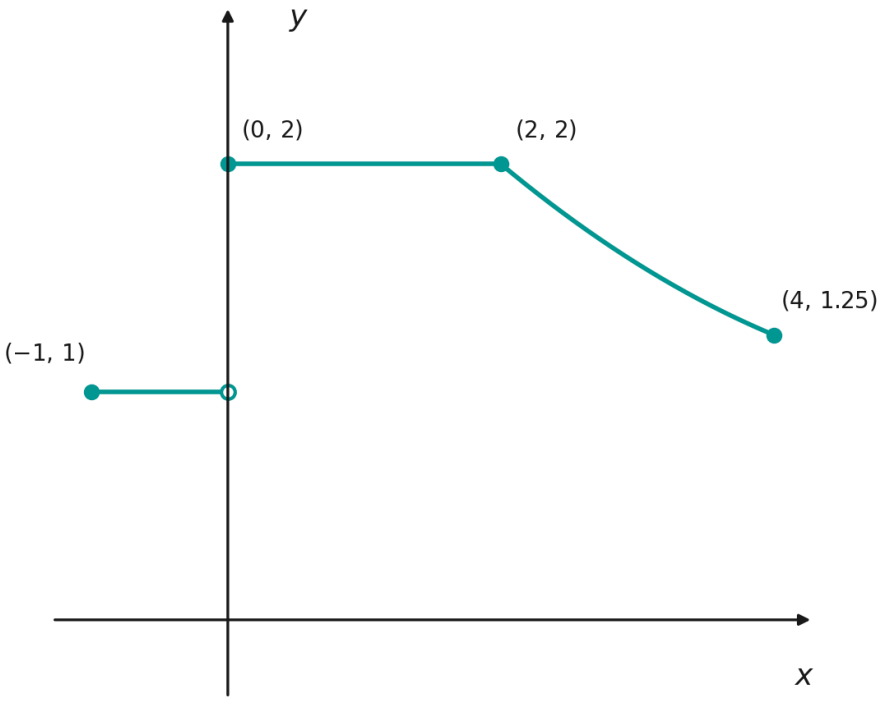
For $1 < \frac{1}{2}x \le 3$, i.e. $2 < x \le 6$: $f\left(\frac{1}{2}x\right) = \frac{1}{4}\left(\frac{1}{2}x - 3\right)^2$, so

$$y = 1 + \frac{1}{4}\left(\frac{1}{2}x - 3\right)^2.$$

At $x = 4$ (the right end of our domain):

$$y = 1 + \frac{1}{4}(2 - 3)^2 = 1.25.$$

Otherwise (in particular for $-1 \le x < 0$): $f\left(\frac{1}{2}x\right) = 0$, so $y = 1$; but at $x = 0$ itself, $y = 1 + f(0) = 2$. As in part **(ii)**, this is a jump: $y \to 1$ as $x \to 0^-$, but $y(0) = 2$.



Question 6 8 marks

Maclaurin series · standard series

Part (i) 8 marks

Since $\ln(1 + u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \ldots$ for $-1 < u \le 1$, we substitute $u = 2x$:

$$\ln(1 + 2x)$$

$$= 2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \dots$$
$$= 2x - 2x^2 + \frac{8}{3}x^3 - \dots$$

$\therefore$ the first three non-zero terms are $2x - 2x^2 + \frac{8}{3}x^3$.

Part (ii)

APPROACH

We expand $ax(1+bx)^c$ by the binomial series for a non-integer index and match coefficients of x, x^2, x^3 with the series found in **(i)**, one equation at a time, rather than trying to solve all three simultaneously.

Using the binomial series for $(1+bx)^c$:

$$(1+bx)^c = 1 + c(bx) + \frac{c(c-1)}{2}(bx)^2 + \frac{c(c-1)(c-2)}{6}(bx)^3 + \dots$$

Multiplying by ax:

$$ax(1+bx)^c = ax + \text{abc}\,x^2 + \frac{ab^2c(c-1)}{2}x^3 + \frac{ab^3c(c-1)(c-2)}{6}x^4 + \dots$$

Comparing the coefficient of x with $2x$:

$$a = 2.$$

Comparing the coefficient of x^2 with $-2x^2$:

$$\text{abc} = -2.$$

Substituting $a = 2$:

$$bc = -1.$$

Comparing the coefficient of x^3 with $\frac{8}{3}x^3$:

$$\frac{ab^2c(c-1)}{2} = \frac{8}{3}$$
$$\Rightarrow ab^2c(c-1) = \frac{16}{3}.$$

Substituting $a = 2$ and $b = -\frac{1}{c}$ (from $bc = -1$):

$$2\left(\frac{1}{c^2}\right)c(c-1) = \frac{16}{3}$$

Simplifying the left side:

$$\frac{2(c-1)}{c} = \frac{16}{3}$$
$$6(c-1) = 16c$$
$$6c - 6 = 16c$$
$$-10c = 6$$
$$c = -\frac{3}{5}.$$

Using $b = -\frac{1}{c}$:

$$b = \frac{5}{3}.$$

$\therefore a = 2,\ b = \frac{5}{3},\ c = -\frac{3}{5}$.

To check your answer: substituting $a = 2,\ b = \frac{5}{3},\ c = -\frac{3}{5}$ back into $\frac{ab^2c(c-1)}{2}$ gives $\frac{8}{3}$, matching the coefficient of x^3 found in **(i)**. ✓

Now we find the coefficient of x^4, using the fourth term of the binomial expansion above:

$$\text{coefficient of } \mathrm{x}^4 = \frac{ab^3c(c-1)(c-2)}{6}.$$

Substituting the values found:

$$= \frac{2\left(\frac{5}{3}\right)^3\left(-\frac{3}{5}\right)\left(-\frac{8}{5}\right)\left(-\frac{13}{5}\right)}{6}$$
$$= -\frac{104}{27}.$$

$\therefore$ the coefficient of x^4 is $-\frac{104}{27}$.

Question 7 10 marks

Vectors · ratio theorem · area

Part (i) 2 marks

Since C lies on OA with $OC : CA = 3 : 2$, C divides OA so that $OC = \frac{3}{5}OA$:

$$\overrightarrow{OC} = \frac{3}{5}\mathbf{a}.$$

Since D lies on OB with $OD : DB = 5 : 6$, D divides OB so that $OD = \frac{5}{11}OB$:

$$\overrightarrow{OD} = \frac{5}{11}\mathbf{b}.$$

Part (ii) 3 marks

APPROACH

A line through two points with known position vectors is written $\mathbf{r} = \mathbf{p} + \lambda(\mathbf{q} - \mathbf{p})$, taking one point as the base and the displacement to the other as the direction. We take B as the base for line BC, so that $\lambda = 0$ gives B and $\lambda = 1$ gives C, matching the form given in the question.

The line BC passes through B (position vector $\mathbf{b}$) in the direction of $C - B$:

$$\mathbf{r} = \mathbf{b} + \lambda\left(\frac{3}{5}\mathbf{a} - \mathbf{b}\right)$$
$$= \frac{3}{5}\lambda\mathbf{a} + (1-\lambda)\mathbf{b},$$

as required (shown).

By the same method, the line AD passes through A (position vector $\mathbf{a}$) in the direction of $D - A$:

$$\begin{aligned}\mathbf{r} &= \mathbf{a} + \mu\left(\frac{5}{11}\mathbf{b} - \mathbf{a}\right) \\ &= (1-\mu)\mathbf{a} + \frac{5}{11}\mu\mathbf{b}.\end{aligned}$$

Part (iii) 5 marks

APPROACH

At the point E where the lines meet, the two expressions for $\mathbf{r}$ are equal for some values of λ and μ. Since $\mathbf{a}$ and $\mathbf{b}$ are non-parallel (non-zero, non-collinear position vectors from O), we can compare coefficients of $\mathbf{a}$ and of $\mathbf{b}$ separately.

At the intersection:

$$\frac{3}{5}\lambda\mathbf{a} + (1-\lambda)\mathbf{b} = (1-\mu)\mathbf{a} + \frac{5}{11}\mu\mathbf{b}.$$

Comparing coefficients of $\mathbf{a}$:

$$\frac{3}{5}\lambda = 1 - \mu. \quad (1)$$

Comparing coefficients of $\mathbf{b}$:

$$1 - \lambda = \frac{5}{11}\mu. \quad (2)$$

From (2):

$$\mu = \frac{11}{5}(1-\lambda).$$

Substituting into (1):

$$\begin{aligned}\frac{3}{5}\lambda &= 1 - \frac{11}{5}(1-\lambda) \\ &= -\frac{6}{5} + \frac{11}{5}\lambda.\end{aligned}$$

Solving:

$$\begin{aligned}\frac{3}{5}\lambda - \frac{11}{5}\lambda &= -\frac{6}{5} \\ -\frac{8}{5}\lambda &= -\frac{6}{5} \\ \lambda &= \frac{3}{4}.\end{aligned}$$

Then:

$$\mu = \frac{11}{5}\left(1 - \frac{3}{4}\right) = \frac{11}{20}.$$

Substituting $\lambda = \frac{3}{4}$ into the equation of line BC:

$$\overrightarrow{OE} = \frac{3}{5}\left(\frac{3}{4}\right)\mathbf{a} + \left(1 - \frac{3}{4}\right)\mathbf{b}$$

$$= \frac{9}{20}\mathbf{a} + \frac{1}{4}\mathbf{b}.$$

To check your answer: substituting $\mu = \frac{11}{20}$ into the equation of line AD gives $\left(1 - \frac{11}{20}\right)\mathbf{a} + \frac{5}{11}\left(\frac{11}{20}\right)\mathbf{b} = \frac{9}{20}\mathbf{a} + \frac{1}{4}\mathbf{b}$, the same position vector. ✓

For the ratio $AE : ED$, let $\mathbf{d} = \overrightarrow{OD} = \frac{5}{11}\mathbf{b}$, so that line AD is $\mathbf{r} = \mathbf{a} + \mu(\mathbf{d} - \mathbf{a})$, where $\mu = 0$ gives A and $\mu = 1$ gives D. Since E corresponds to $\mu = \frac{11}{20}$,

$$\overrightarrow{AE} = \frac{11}{20}(\mathbf{d} - \mathbf{a}), \qquad \overrightarrow{ED} = \left(1 - \frac{11}{20}\right)(\mathbf{d} - \mathbf{a}) = \frac{9}{20}(\mathbf{d} - \mathbf{a}),$$

so $\overrightarrow{AE}$ and $\overrightarrow{ED}$ are parallel and in the ratio of these scalars:

$$AE : ED = \frac{11}{20} : \frac{9}{20} = 11 : 9.$$

$\therefore \overrightarrow{OE} = \frac{9}{20}\mathbf{a} + \frac{1}{4}\mathbf{b}$ and $AE : ED = 11 : 9$.

Question 8 11 marks

Sequences and series · AP and GP in context

Since each lap is $400\ m$ and 50 laps make up the $20\ km$ run, the required total time lies between $1\frac{1}{2}$ hours and $1\frac{3}{4}$ hours inclusive, i.e.

$$5400\ s \leq \text{total time} \leq 6300\ s.$$

Part (i) 4 marks

Athlete A's lap times form an arithmetic progression with first term T and common difference 2, so the total time for 50 laps is

$$S_{50} = \frac{50}{2}(2T + 49(2))$$
$$= 50T + 2450.$$

Using the required time interval:

$$5400 \leq 50T + 2450 \leq 6300.$$

Solving:

$$2950 \leq 50T \leq 3850$$
$$59 \leq T \leq 77.$$

$\therefore T$ satisfies $59\ s \leq T \leq 77\ s$.

Part (ii) 4 marks

> **APPROACH**
> "2% more than the time for the previous lap" makes each lap time 1.02 times the one before, so the lap times form a geometric progression with common ratio 1.02, not an arithmetic one.

Athlete B's lap times form a geometric progression with first term t and common ratio 1.02, so the total time for 50 laps is

$$S_{50} = \frac{t\left(1.02^{50} - 1\right)}{1.02 - 1}.$$

By GC, $1.02^{50} = 2.69159\ldots$, so

$$S_{50} = 84.5794\ldots t.$$

Using the required time interval:

$$5400 \leq 84.5794\ldots t \leq 6300.$$

Solving:

$$63.8 \leq t \leq 74.5 \ \ (3\ s.f.).$$

$\therefore t$ satisfies $63.8\ s \leq t \leq 74.5\ s$ (to 3 s.f.).

Note: $84.5794\ldots t$ is increasing in t, so the direction of both inequalities is unchanged when dividing through by it.

Part (iii) 3 marks

> **APPROACH**
> "Completes the run in exactly $1\frac{1}{2}$ hours" fixes $S_{50} = 5400$ for each athlete, which pins down T and t at the lower end of the ranges found in **(i)** and **(ii)**. We then find each athlete's 50th lap time directly from their progression.

For athlete A, setting $S_{50} = 5400$ in $50T + 2450 = 5400$ gives $T = 59$ (the lower bound found in **(i)**). The 50th lap time is the 50th term of the arithmetic progression:

$$T + 49(2) = 59 + 98 = 157\ s.$$

For athlete B, setting $S_{50} = 5400$ gives $t = 63.845\ldots$ (the lower bound found in **(ii)**, kept unrounded here). The 50th lap time is the 50th term of the geometric progression:

$$t \times 1.02^{49}.$$

By GC,

$$63.845\ldots \times 1.02^{49} = 168.48\ldots\ s.$$

The difference in the athletes' 50th lap times is

$$168.48\ldots - 157 = 11.48\ldots\ s.$$

$\therefore$ the difference is $11\ s$ (to the nearest second), with athlete B's 50th lap taking longer.

To check your answer: $84.5794\ldots \times 63.845\ldots = 5400$, confirming t was recovered correctly from $S_{50} = 5400$. ✓

Question 9 13 marks

Complex numbers · conjugates · polar form

Part (a) 5 marks

APPROACH

"Purely imaginary" means the real part of $\frac{w^2}{w^*}$ is zero (with the imaginary part non-zero, which we check afterwards). To split $\frac{w^2}{w^*}$ into real and imaginary parts we multiply numerator and denominator by w, the conjugate of w^*.

Let $w = a + ib$, where a and b are non-zero real numbers, so $w^* = a - ib$.

$$\begin{aligned} w^2 &= (a+ib)^2 \\ &= a^2 - b^2 + 2\text{abi}. \end{aligned}$$

$$\begin{aligned} \frac{w^2}{w^*} &= \frac{\left(a^2 - b^2 + 2\text{abi}\right)(a+ib)}{(a-ib)(a+ib)} \\ &= \frac{a^3 + 3a^2bi - 3ab^2 - b^3 i}{a^2+b^2} \\ &= \frac{a^3 - 3ab^2}{a^2+b^2} + i\,\frac{3a^2b - b^3}{a^2+b^2}. \end{aligned}$$

Since $\frac{w^2}{w^*}$ is purely imaginary, its real part is zero:

$$\frac{a^3 - 3ab^2}{a^2+b^2} = 0.$$

Since $a^2 + b^2 \neq 0$ (as a, b are non-zero real numbers):

$$\begin{aligned} a^3 - 3ab^2 &= 0 \\ \Rightarrow a\left(a^2 - 3b^2\right) &= 0. \end{aligned}$$

Since $a \neq 0$:

$$\begin{aligned} a^2 - 3b^2 &= 0 \\ \Rightarrow b^2 &= \frac{a^2}{3} \\ \Rightarrow b &= \pm\frac{a}{\sqrt{3}} = \pm\frac{a\sqrt{3}}{3}. \end{aligned}$$

Note: substituting $b^2 = \frac{a^2}{3}$ into the imaginary part gives $3a^2b - b^3 = b\left(3a^2 - b^2\right) = \frac{8a^2}{3}b$, which is non-zero for $a, b \neq 0$, so $\frac{w^2}{w^*}$ is genuinely purely imaginary (not zero) for these values.

$\therefore w = a + \frac{a\sqrt{3}}{3}i$ or $w = a - \frac{a\sqrt{3}}{3}i$.

Part (b)(i) 4 marks

APPROACH

Write $-32i$ in modulus-argument form and apply De Moivre's theorem to extract the five fifth roots, choosing the representative for each argument that lies in $(-\pi, \pi]$.

Since $-32i = 32\left(\cos\left(-\frac{1}{2}\pi\right) + i\sin\left(-\frac{1}{2}\pi\right)\right)$, the equation $z^5 = -32i$ gives

$$z^5 = 32\left(\cos\left(-\frac{1}{2}\pi\right) + i\sin\left(-\frac{1}{2}\pi\right)\right).$$

By De Moivre's theorem, the five roots are

$$z = 32^{\frac{1}{5}}\left(\cos\left(\frac{-\frac{1}{2}\pi + 2k\pi}{5}\right) + i\sin\left(\frac{-\frac{1}{2}\pi + 2k\pi}{5}\right)\right), \quad k = 0, \pm 1, \pm 2,$$

so each z has modulus $32^{\frac{1}{5}} = 2$ and argument $-\frac{\pi}{10} + \frac{2k\pi}{5}$.

Taking $k = 0, \pm 1, \pm 2$ gives the five arguments, all already in $(-\pi, \pi]$:

$$\arg z = -\frac{9\pi}{10}, -\frac{\pi}{2}, -\frac{\pi}{10}, \frac{3\pi}{10}, \frac{7\pi}{10}.$$

$\therefore$ each value of z has modulus 2, with arguments $-\frac{9\pi}{10}, -\frac{\pi}{2}, -\frac{\pi}{10}, \frac{3\pi}{10}, \frac{7\pi}{10}$.

Part (b)(ii) 4 marks

APPROACH

Identify z_1 and z_2 from the argument ranges given, then combine them using the sum-to-product identities for $\cos A - \cos B$ and $\sin A + \sin B$: this puts $z_1 - z_2$ directly into modulus-argument form, giving both the argument and the modulus in one calculation.

Since $\frac{1}{2}\pi < \arg z_1 < \pi$, we take $\arg z_1 = \frac{7\pi}{10}$, so $z_1 = 2\left(\cos\frac{7\pi}{10} + i\sin\frac{7\pi}{10}\right)$.

Since $-\pi < \arg z_2 < -\frac{1}{2}\pi$, we take $\arg z_2 = -\frac{9\pi}{10}$, so $z_2 = 2\left(\cos\frac{9\pi}{10} - i\sin\frac{9\pi}{10}\right)$, using $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$.

$$z_1 - z_2 = 2\left(\cos\frac{7\pi}{10} - \cos\frac{9\pi}{10}\right) + 2i\left(\sin\frac{7\pi}{10} + \sin\frac{9\pi}{10}\right).$$

Using $\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$ and $\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$, with $A = \frac{7\pi}{10}$, $B = \frac{9\pi}{10}$, so $\frac{A+B}{2} = \frac{4\pi}{5}$ and $\frac{A-B}{2} = -\frac{\pi}{10}$:

$$\begin{aligned} z_1 - z_2 &= 4\sin\frac{4\pi}{5}\sin\frac{\pi}{10} + 4i\sin\frac{4\pi}{5}\cos\frac{\pi}{10} \\ &= 4\sin\frac{4\pi}{5}\left(\sin\frac{\pi}{10} + i\cos\frac{\pi}{10}\right). \end{aligned}$$

Since $\sin\frac{\pi}{10} = \cos\left(\frac{1}{2}\pi - \frac{\pi}{10}\right) = \cos\frac{2\pi}{5}$ and $\cos\frac{\pi}{10} = \sin\left(\frac{1}{2}\pi - \frac{\pi}{10}\right) = \sin\frac{2\pi}{5}$, and $\sin\frac{4\pi}{5} = \sin\left(\pi - \frac{4\pi}{5}\right) = \sin\frac{\pi}{5} > 0$:

$$z_1 - z_2 = 4\sin\frac{\pi}{5}\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right).$$

This is already in the form $r(\cos\alpha + i\sin\alpha)$ with $r = 4\sin\frac{\pi}{5} > 0$, so we read off the modulus and argument directly.

$\therefore \arg(z_1 - z_2) = \frac{2\pi}{5}$, and $|z_1 - z_2| = 4\sin\frac{\pi}{5}$. (shown)

Question 10 12 marks

Integration · area · volume of revolution

Part (i) 4 marks

APPROACH

From the diagram, A_1 is bounded below by the x-axis, on the left by $y = \sin x$ (from O to P) and on the right by $y = \cos x$ (from P to $\left(\frac{1}{2}\pi,\ 0\right)$); A_2 is bounded on the left by the y-axis, above by $y = \cos x$ (from $(0,\ 1)$ to P) and below by $y = \sin x$ (from O to P). Both areas split naturally at $x = \frac{1}{4}\pi$, the x-coordinate of P.

$$A_1 = \int_0^{\frac{1}{4}\pi} \sin x\,dx + \int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \cos x\,dx$$

$$= \left[-\cos x\right]_0^{\frac{1}{4}\pi} + \left[\sin x\right]_{\frac{1}{4}\pi}^{\frac{1}{2}\pi}$$

$$= \left(-\frac{1}{2}\sqrt{2} + 1\right) + \left(1 - \frac{1}{2}\sqrt{2}\right)$$

$$= 2 - \sqrt{2}.$$

$$A_2 = \int_0^{\frac{1}{4}\pi} (\cos x - \sin x)\,dx$$

$$= \left[\sin x + \cos x\right]_0^{\frac{1}{4}\pi}$$

$$= \left(\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}\right) - (0 + 1)$$

$$= \sqrt{2} - 1.$$

$$\frac{A_1}{A_2} = \frac{2 - \sqrt{2}}{\sqrt{2} - 1}$$

$$= \frac{\sqrt{2}\left(\sqrt{2} - 1\right)}{\sqrt{2} - 1}$$

$$= \sqrt{2}. \quad \text{(shown)}$$

Part (ii) 8 marks

APPROACH

The region described has $x = 0$ as its left boundary throughout, so rotating about the y-axis gives a solid of revolution by the disc method, with radius equal to x expressed as a function of y.

For $0 \le x \le \frac{1}{4}\pi$, $y = \sin x$ is increasing, so it has an inverse there; writing x in terms of y,

$$x = \sin^{-1} y.$$

The region bounded by $y = \sin x$ between O and P, the line $y = \frac{1}{2}\sqrt{2}$, and the y-axis is the set of points with $0 \le y \le \frac{1}{2}\sqrt{2}$ and $0 \le x \le \sin^{-1} y$. Rotating this about the y-axis through $360°$, each cross-section at height y is a disc of radius $x = \sin^{-1} y$, so the volume is

$$V = \pi \int_0^{\frac{1}{2}\sqrt{2}} x^2 \, dy$$

$$= \pi \int_0^{\frac{1}{2}\sqrt{2}} \left(\sin^{-1} y\right)^2 dy. \quad \text{(shown)}$$

Part (iii)

APPROACH

The substitution $y = \sin u$ removes the inverse sine, since $\sin^{-1} y = u$ once u is restricted to the range where $y = \sin u$ has an inverse; the limits transform along with the variable.

Let $y = \sin u$, so $dy = \cos u \, du$. For $u \in \left[0, \ \frac{1}{4}\pi\right] \subset \left[-\frac{1}{2}\pi, \ \frac{1}{2}\pi\right]$, the range of the principal branch of $\sin^{-1}$, we have $\sin^{-1} y = \sin^{-1}(\sin u) = u$.

When $y = 0$, $u = 0$; when $y = \frac{1}{2}\sqrt{2}$, $\sin u = \frac{1}{2}\sqrt{2}$, so $u = \frac{1}{4}\pi$.

$$\pi \int_0^{\frac{1}{2}\sqrt{2}} \left(\sin^{-1} y\right)^2 dy = \pi \int_0^{\frac{1}{4}\pi} u^2 \cos u \, du.$$

$\therefore a = 0$, $b = \frac{1}{4}\pi$.

APPROACH

$\int u^2 \cos u \, du$ needs integration by parts twice, differentiating u^2 down to a constant while integrating $\cos u$ each time.

$$\int u^2 \cos u \, du = u^2 \sin u - \int 2u \sin u \, du$$

$$= u^2 \sin u - 2\left(-u \cos u + \int \cos u \, du\right)$$

$$= u^2 \sin u + 2u \cos u - 2 \sin u \ (+C).$$

$$\pi \int_0^{\frac{1}{4}\pi} u^2 \cos u \, du = \pi \left[u^2 \sin u + 2u \cos u - 2 \sin u\right]_0^{\frac{1}{4}\pi}$$

$$= \pi \left(\frac{\pi^2}{16} \cdot \frac{1}{2}\sqrt{2} + 2 \cdot \frac{1}{4}\pi \cdot \frac{1}{2}\sqrt{2} - 2 \cdot \frac{1}{2}\sqrt{2}\right)$$

$$= \pi \left(\frac{\sqrt{2}\pi^2}{32} + \frac{\sqrt{2}\pi}{4} - \sqrt{2}\right)$$

$$= \frac{\sqrt{2}\pi^3}{32} + \frac{\sqrt{2}\pi^2}{4} - \sqrt{2}\pi.$$

$\therefore$ the exact volume is $\frac{\sqrt{2}\pi^3}{32} + \frac{\sqrt{2}\pi^2}{4} - \sqrt{2}\pi$, i.e. $\sqrt{2}\pi\left(\frac{\pi^2}{32} + \frac{\pi}{4} - 1\right)$.

Question 11 15 marks

Parametric equations · turning points · area

Part (i) 3 marks

Since $x = \sin^3\theta$ and $y = 3\sin^2\theta\cos\theta$,

$$\frac{dx}{d\theta} = 3\sin^2\theta\cos\theta.$$

$$\begin{aligned}\frac{dy}{d\theta} &= 3\left(2\sin\theta\cos\theta\cdot\cos\theta + \sin^2\theta\cdot(-\sin\theta)\right)\\ &= 6\sin\theta\cos^2\theta - 3\sin^3\theta.\end{aligned}$$

By the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta}\\ &= \frac{6\sin\theta\cos^2\theta - 3\sin^3\theta}{3\sin^2\theta\cos\theta}\\ &= \frac{2\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta}\\ &= 2\cot\theta - \tan\theta. \quad \text{(shown)}\end{aligned}$$

Part (ii) 6 marks

> **APPROACH**
>
> A turning point occurs where $\frac{dy}{dx} = 0$. We solve $2\cot\theta = \tan\theta$ for $\tan\theta$ directly, since the required form $\tan\theta = \sqrt{k}$ asks for it.

$$\begin{aligned}&2\cot\theta = \tan\theta\\ \Rightarrow\ &\frac{2\cos\theta}{\sin\theta} = \frac{\sin\theta}{\cos\theta}\\ \Rightarrow\ &2\cos^2\theta = \sin^2\theta.\end{aligned}$$

Dividing both sides by $\cos^2\theta$, which is non-zero since $0 \le \theta \le \frac{1}{2}\pi$ excludes $\theta = \frac{1}{2}\pi$ at a turning point (there $\frac{dx}{d\theta} = 0$ too, so it is not a genuine turning point of the curve):

$$2 = \tan^2\theta.$$

Since $0 \le \theta \le \frac{1}{2}\pi$ gives $\tan\theta \ge 0$, we take the positive root.

$\therefore \tan\theta = \sqrt{2}$, so $k = 2$.

> **APPROACH**
>
> With $\tan\theta = \sqrt{2}$, we read $\sin\theta$ and $\cos\theta$ off a right triangle with opposite $\sqrt{2}$, adjacent 1, hypotenuse $\sqrt{3}$, then substitute into x and y.

Since $\tan\theta = \sqrt{2}$, $\sin\theta = \frac{\sqrt{2}}{\sqrt{3}}$ and $\cos\theta = \frac{1}{\sqrt{3}}$.

$$x = \sin^3\theta = \left(\frac{\sqrt{2}}{\sqrt{3}}\right)^3$$
$$= \frac{2\sqrt{2}}{3\sqrt{3}}$$
$$= \frac{2\sqrt{6}}{9}.$$
$$y = 3\sin^2\theta\cos\theta = 3 \cdot \frac{2}{3} \cdot \frac{1}{\sqrt{3}}$$
$$= \frac{2}{\sqrt{3}}$$
$$= \frac{2\sqrt{3}}{3}.$$

$\therefore$ the turning point is $\left(\frac{2\sqrt{6}}{9}, \frac{2\sqrt{3}}{3}\right)$.

Explanation: $2\cot\theta$ and $-\tan\theta$ are both strictly decreasing on $\left(0, \frac{1}{2}\pi\right)$, so $\frac{dy}{dx} = 2\cot\theta - \tan\theta$ is strictly decreasing in θ, changing sign from positive to negative exactly once, at $\tan\theta = \sqrt{2}$. Since $\frac{dx}{d\theta} = 3\sin^2\theta\cos\theta \geq 0$ throughout $0 \leq \theta \leq \frac{1}{2}\pi$, x increases as θ increases, so $\frac{dy}{dx}$ also changes from positive to negative as x increases through this point.

$\therefore$ the turning point is a maximum.

Part (iii) 3 marks

APPROACH

As θ increases from 0 to $\frac{1}{2}\pi$, $x = \sin^3\theta$ increases monotonically from 0 to 1 (since $\frac{dx}{d\theta} \geq 0$), and $y \geq 0$ throughout, so the area under C can be written as $\int y\,dx$ over this range and converted to an integral in θ.

$$\text{Area} = \int_0^1 y\,dx = \int_0^{\frac{1}{2}\pi} y\frac{dx}{d\theta}\,d\theta$$
$$= \int_0^{\frac{1}{2}\pi} 3\sin^2\theta\cos\theta \cdot 3\sin^2\theta\cos\theta\,d\theta$$
$$= \int_0^{\frac{1}{2}\pi} 9\sin^4\theta\cos^2\theta\,d\theta. \quad \text{(shown)}$$

By GC, Area $=0.884$ (3 d.p.).

Note: this integral has the exact closed form $\frac{9\pi}{32}$, since $0.884 \approx \frac{9\pi}{32} = 0.8836\ldots$; the question only asks for the 3 d.p. calculator value.

Part (iv) 3 marks

APPROACH

Substitute $y = ax$ directly into the parametric equations of C to find where the line meets the curve, then divide out the root $\theta = 0$ (which gives the origin) to isolate $\tan\theta$.

At a point of intersection, $y = ax$ gives

$$3\sin^2\theta\cos\theta = a\sin^3\theta.$$

For $\theta \neq 0$ (which gives the origin), dividing both sides by $\sin^2\theta$, which is non-zero:

$$3\cos\theta = a\sin\theta$$
$$\Rightarrow \tan\theta = \frac{3}{a}. \quad \text{(shown)}$$

For the line to pass through the maximum point found in **(ii)**, where $\tan\theta = \sqrt{2}$:

$$\frac{3}{a} = \sqrt{2}$$
$$\Rightarrow a = \frac{3}{\sqrt{2}}$$
$$\Rightarrow a = \frac{3\sqrt{2}}{2}.$$

$\therefore a = \frac{3\sqrt{2}}{2}$.

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